Three Body Problems in Evolutionary Game Dynamics: Convergence, Periodicity and Limit Cycles

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Abstract

We study the asymptotic behavior of replicator dynamics in settings of network interaction. We focus on three agent graphical games where each edge/game is either a 2x2 zero-sum or a 2x2 coordination game. Using tools from dynamical systems such as Lyapunov functions and invariant functions we establish that this simple family of games can exhibit an interesting range of behaviors such as global convergence, periodicity for all initial conditions as well as limit cycles. In contrast, we do not observe more complex behavior such as toroids or chaos whilst it is possible to reproduce them in slightly more complicated settings.

1 Introduction

The analysis of multi-agent system dynamics is a central question for numerous fields including AI, game theory as well as systems engineering. Despite the undoubtable importance of such questions identifying a clear path towards analytical success has been tricky and numerous different approaches have been proposed and explored Shoham et al. (2007); Vohra and Wellman (2007); Stone (2007).

What makes the analysis of MAL systems so inherently elusive is that multi-agent systems from their very nature allow the emergence of rather complex patterns of behavior Gabbai et al. (2005). Even when it comes to simplified evolutionary game theory models of adaptive agents behavior even relatively simple systems based on variants of Rock, Paper, Scissors games can lead to chaotic dynamics Sato et al. (2002).

The reason behind this emergence of complexity in evolutionary dynamics has to do with the building components of these systems: i.e. the dynamics themselves. Replicator dynamics is arguably the most well known and extensively studied evolutionary dynamic Weibull (1995); Hofbauer and Sigmund (1998); Sandholm (2010). It is a continuous time dynamic that is the smooth analogue of the well known Multiplicative Weights Update algorithm Arora et al. (2012); Kleinberg et al. (2009). Replicator dynamics when applied to multi-agent games results in nonlinear dynamical systems whose behavior in three of more dimensions can be chaotic. Examples of complex recurrent behavior have been

shown to emerge in replicator systems with four or more independent variables Skyrms (1992); Sato et al. (2002); Chawanya (1996); Piliouras and Shamma (2014). In contrast, replicator dynamics in two agent, two strategy games (which have only two independent variables) are known to have simple limit behaviors, either convergence to equilibria or periodicity Eshel and Akin (1983); Hofbauer and Sigmund (1998); Papadimitriou and Piliouras (2016). This leaves an interesting not well understood gap about the possible behaviors of three dimensional replicator dynamics.

Within this context, we focus on a rather natural and archetypal class of three player games. We consider three player graphical games, where each player corresponds to a vertex of a triangle and each edge corresponds to a two by two game (triangular game). Furthermore, we focus on the case where each edge game is either completely adversarial (zero-sum game) or common utility (coordination/partnership) game. Intuitively, one can think of these network interactions as encoding friend-or-foe type of relationships, where coordination games correspond to perfectly aligned interests (friend) whereas zero-sum games correspond to perfectly misaligned interests (foe).

(a) z-z-z game (b) c-c-c game

(c) c-c-z game (d) z-z-c game

Figure 1: All types of 3-player triangular games.

Our Results and Techniques. We analyze the behavior of replicator dynamics in the setting of friend-or-foe triangles. In the case of all zero-sum interactions **(z-z-z)**, we prove that as long as the game has a fully mixed Nash equilibrium then all system trajectories are perfectly periodic. If the game does not have an interior equilibrium then all interiors initial conditions converge to the boundary. The key technical observation is that after a change of variables the systems trajectories are shown to be planar (i.e. two dimensional). Combining this result with techniques for proving more complicated types of recurrence for replicator dynamics in zero-sum networks we establish the periodicity of the trajectories. Our technique for proving invariant functions for replicator dynamics is quite generic and can be applied to large networks of friend-or-foe games. This includes the case of **(z-c-c)** games. For this class of games, we can show that the dynamics are still planar and hence due to the Poincaré-Bendixson theorem these systems provably cannot exhibit chaos. Boundary limit cycles, interior periodic orbits as well as convergence to equilibria are all shown to be possible. The case of **(c-c-c)** is relatively straightforward as the corresponding three player games are potential games and convergence to equilibria is guaranteed from all initial conditions. Finally, the case of **(z-z-c)** we show experimentally

that it can exhibit limit behavior that is not observed in any of other settings including boundary limit cycles where the two collaborating (friend) agents and the single agent without any friends take turns best responding simultaneously to each other. In this case, the dynamics can result in the evolutionary formation of a team of two cooperating agents that act in unison against the single opposing agent, which leads to limit cycles.

2 Preliminaries

2.1 Separable polymatrix multiplayer game

A graphical polymatrix game is defined via an undirected graph $G = (V, E)$, where V corresponds to the set of agents of the game and where every edge corresponds to a bimatrix game between its two endpoints/agents. We denote by S_i the set of strategies of agent i. We denote the bimatrix game on edge $(i, k) \in E$ via a pair of payoff matrices: $A^{i,k}$ of dimension $|S_i| \times |S_k|$ and $A^{k,i}$ of dimension $|S_k| \times |S_i|$. Let $s \in \times_i S_i$ be a strategy profile of the game. We denote by $s_i \in S_i$ the respective strategy of agent i. The payoff of agent $i \in V$ in strategy profile s is equal to the sum of the payoffs that agent i receives from all the bimatrix games she participates in. Specifically, $u_i(s) = \sum_{(i,k)\in E} A^{i,k}_{s_i,s_k}$. In addition, the social welfare of a joint strategy s is defined as $SW(s) = \sum_i u_i(s)$.

A randomized strategy x for agent i lies on the simplex $\Delta(S_i) = \{p \in \mathbb{R}_+^{|S_i|} : \sum_i p_i = 1\}$. A randomized strategy x is said to be fully mixed if it lies in the interior of the simplex, *i.e.* if $x_i > 0$ for all strategies $i \in S_i$.

A (mixed) Nash equilibrium is a profile of mixed strategies such that no agent can improve her (expected) payoff by unilaterally deviating to another strategy.

2.2 Replicator Dynamics

The replicator equation is commonly used to describe game dynamics of learning agents in evolutionary game theory. In its continuous form it is give by the following differential equation:

$$\dot{x}_i \triangleq x_i[u_i(x) - \hat{u}(x)], \quad \hat{u}(x) = \sum_{i=1}^{n} x_i u_i(x)$$

where x_i is the proportion of type i in the population, $x = (x_1, \ldots, x_m)$ is the vector of the distribution of types in the population, $u_i(x)$ is the fitness of type i, and $\hat{u}(x)$ is the average population fitness. The state vector x can also be interpreted as a randomized strategy of an adaptive agent that learns to optimize over its possible actions given an online stream of payoff vectors.

An interior point of the state space is a fixed point for the replicator if and only if it is a fully mixed Nash equilibrium of the game. The interior (the boundary) of the state space $\times_i \Delta(S_i)$ are invariants for the replicator.

We write down the replicator dynamics for the generic multiplayer game in a compact form as follows:

$$\dot{x}_{iR} = x_{iR}\big(u^i(R) - \sum_{R' \in S_i} x_{iR'} u^i(R')\big)$$

for each agent $i \in N$, action $R \in S_i$, and where we define $u^i(R) = \mathrm{E}_{s_{-i} \sim x_{-i}} u_i(R, s_{-i})$.

2.3 Dynamical Systems Theory

In this section we provide a quick introduction to the main ideas in the topology of dynamical systems and certain important theorems that we will be hinging upon to perform the analysis of the three player replicator systems and the book by Bhatia and Szegö Bhatia and Szegö (2006) serves as a good reference (see also Weibull Weibull (1995)).

Since our state space is compact and the replicator vector field is Lipschitz-continuous, we can present the unique solution of our ordinary differential equation as a continuous map $\Phi : \mathcal{S} \times \mathbb{R} \to \mathcal{S}$ called flow of the system. Fixing starting point $x \in \mathcal{S}$ defines a function of time which captures the trajectory (orbit, solution path) of the system with the given starting point. This corresponds to the graph of $\Phi(x, \cdot) : \mathbb{R} \to \mathcal{S}$, *i.e.*, the set $\{(t, y) : y = \Phi(x, t) \text{ for some } t \in \mathbb{R}\}$. The trajectory captures the evolution of the state of a system given an initial starting point.

A central concept in dynamical systems is the notion of trajectory (or path) through a state $x \in \mathcal{S}$. This corresponds to the graph of the flow $\Phi(x, \cdot) : \mathbb{R} \to \mathcal{S}$, *i.e.* the set $\{(t, y) : y = \Phi(x, t) \text{ for some } t \in \mathbb{R}\}$. If the starting point x does not correspond to an equilibrium, then we wish to capture the asymptotic behavior of the system (informally the limit of $\Phi(x, t)$ when t goes to infinity). A function f between two topological spaces is called a *homeomorphism* if it has the following properties: f is a bijection, f is continuous, and f has a continuous inverse. A function f between two topological spaces is called a *diffeomorphism* if it has the following properties: f is a bijection, f is continuously differentiable, and f has a continuously differentiable inverse.

Lemma 1. *(Topological conjugacy) Two flows $\Phi^t : A \to A$ and $\Psi^t : B \to B$ are conjugate if there exists a homeomorphism $g : A \to B$ such that for each $x \in A$ and $t \in \mathbb{R}$: $g(\Phi^t(x)) = \Psi^t(g(x))$.*

Furthermore, two flows $\Phi^t : A \to A$ and $\Psi^t : B \to B$ are *diffeomorhpic* if there exists a diffeomorphism $g : A \to B$ such that for each $x \in A$ and $t \in \mathbb{R}$ $g(\Phi^t(x)) = \Psi^t(g(x))$. If two flows are diffeomorphic, then their vector fields are related by the derivative of the conjugacy. That is, we get precisely the same result that we would have obtained if we simply transformed the coordinates in their differential equations Meiss (2007).

Theorem 2. ***Poincaré Recurrence*** *Poincaré (1890); Barreira (2006)If a flow preserves volume and has only bounded orbits then for each open set there exist orbits that intersect the set infinitely often.*

The Poincaré-Bendixson theorem implies that chaos in not possible in planar dynamical systems in continuous time.

Theorem 3. ***Poincaré-Bendixson theorem*** *Bendixson (1901); Teschl (2012) Given a differentiable real dynamical system defined on an open subset of the plane, then every non-empty compact ω-limit set of an orbit, which contains only finitely many fixed points, is either a fixed point, a periodic orbit, or a connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.*

Lyapunov Function: A Lyapunov (or potential) function $V : \mathcal{S} \to \mathbb{R}$ is a function that strictly decreases along every non-trivial trajectory of the dynamical system. For continuous time dynamical systems it holds that $\frac{dV}{dt} \leq 0$ with equality attained only at the equilibrium of the system. For more information see Khalil (1996).

2.4 Information Theory

Entropy is a measure of the uncertainty of a random variable and captures the expected information value from a measurement of the random variable. The entropy H of a discrete random variable X with possible values $\{1, \ldots, n\}$ and probability mass function $p(X)$ is defined as $H(X) = -\sum_{i=1}^{n} p(i) \ln p(i)$.

Given two probability distributions p and q of a discrete random variable their *K-L divergence* (relative entropy) is defined as $D_{\mathrm{KL}}(p\|q) = \sum_i \ln\left(\frac{p(i)}{q(i)}\right) p(i)$. It is the average of the logarithmic difference between the probabilities p and q, where the average is taken using the probabilities p.

For more details the reader should refer to the classic text by Cover and Thomas Cover and Thomas (2012).

3 Global Analysis of Three Player Games

In this section, we present the global analysis of all possible three player games with two strategies each, by using different tools from dynamical systems theory. Before looking at the specific sub-cases let us consider a specific structure of the polymatrix game and show the existence of an invariant function which will help us to argue about the global dynamics in the specific sub-cases (**(z-z-z)** and **(c-c-z)**). We also present simulation results along side the different cases, with different forms of the classic matching pennies that is used for playing a zero-sum and a coordination game. To make the notations simpler, we define the following shorthand:

$$MP^{g}_{i,j}(a, b, c, d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

which means that players i, j play a game with payoff matrices as specified and the game type is g, where $g = Z$ is a zero-sum game and $g = C$ is a coordination game. So the standard matching pennies between players i, j would be given by $MP^{Z}_{i,j}(1, -1, -1, 1)$.

3.1 Invariant function for Zero-Coordination-Zero Bipartite Games

We generalise the results obtained in Piliouras and Shamma (2014) for a network zero-sum game and in Panageas and Piliouras (2016) for bipartite full coordination games to a more generic bipartite graph (where connections within the partitions are allowed) with the players within the disjoints sets are playing a zero-sum game between them and the players between the disjoint sets are playing coordination games with one another. So, we define the disjoint set of vertices to be V_L and V_R ($V_L \cap V_R = \phi$ and $V_L \cup V_R = V$), with players $i, j \in V_L$ playing a seperable zero-sum game, while players in $i \in V_L$, $j \in V_R$ are playing coordination games. Let E_L consists of the edges whose end points are players in V_L playing a zero-sum game and similarly define E_R for V_R. E_C consists of player $i \in V_L$ and $j \in V_R$, where i, j are playing a coordination game. Then,

Theorem 4. *Let Φ denote the flow of the replicator dynamic when applied to a zero-coordination-zero bipartite game that has an interior (i.e. fully mixed) Nash equilibrium q then given any (interior) starting point $x_0 \in \times_i \Delta(S_i)$, the cross entropy between q and the state of system $\Phi(x_0, t)$ is a constant of the motion, i.e., it remains constant as we move along any system trajectory.*

Proof The support of the state of system (*e.g.,* the strategies played with positive probability) is an invariant of the flow, so it suffices to prove this statement for each starting point x_0 at time $t = 0$. We examine the time derivative of $H(q, \Phi(x_0, t)) = -\left(\sum_{i\in V_L}\sum_{R\in S_i} q_{iR}\ln(x_{iR}) - \sum_{j\in V_R}\sum_{R\in S_j} q_{jR}\ln(x_{jR})\right)$. Consider the time derivative of the function as follows:

$$
\begin{aligned}
&\frac{dH(q,\Phi(x_0,t))}{dt} = \sum_{i\in V_L}\sum_{R\in S_i} q_{iR}\frac{d\ln(x_{iR})}{dt} - \sum_{j\in V_R}\sum_{R\in S_j} q_{jR}\frac{d\ln(x_{jR})}{dt} \\
&= \sum_{i\in V_L}\sum_{R\in S_i} q_{iR}\frac{\dot{x}_{iR}}{x_{iR}} - \sum_{j\in V_R}\sum_{R\in S_j} q_{jR}\frac{\dot{x}_{jR}}{x_{jR}} \\
&= \sum_{i\in V_L}\sum_{(i,m)\in E_L}\sum_{(i,k)\in E_C} q_i^T\left(A^{i,m}x_m + A^{i,k}x_k\right) - x_i^T\left(A^{i,m}x_m + A^{i,k}x_k\right) \\
&- \left(\sum_{j\in V_R}\sum_{(j,n)\in E_R}\sum_{(j,k)\in E_C} q_j^T\left(A^{i,n}x_n + A^{j,k}x_k\right) - x_j^T\left(A^{j,n}x_m + A^{j,k}x_k\right)\right) \\
&= \sum_{i\in V_L}\sum_{(i,m)\in E_L} q_i^T A^{i,m}x_m - x_i^T A^{i,m}x_m + \sum_{i\in V_L}\sum_{(i,k)\in E_C} q_i^T A^{i,k}x_k - x_i^T A^{i,k}x_k \\
&- \sum_{j\in V_R}\sum_{(j,k)\in E_C} q_j^T A^{j,k}x_k - x_j^T A^{j,k}x_k - \sum_{j\in V_L}\sum_{(j,n)\in E_R} q_j^T A^{j,n}x_n - x_j^T A^{j,n}x_n \qquad (2) \\
&= \sum_{i\in V_L}\sum_{(i,m)\in E_L}\left(q_i^T - x_i^T\right)A^{i,m}x_m + \sum_{i\in V_L}\sum_{(i,k)\in E_C}\left(q_i^T - x_i^T\right)A^{i,k}x_k \\
&- \sum_{j\in V_R}\sum_{(j,k)\in E_C}\left(q_j^T - x_j^T\right)A^{j,k}x_k - \sum_{j\in V_R}\sum_{(j,n)\in E_R}\left(q_j^T - x_j^T\right)A^{j,n}x_n \\
&= \sum_{i\in V_L}\sum_{E_L=(i,m)}\left(q_i^T - x_i^T\right)A^{i,m}\left(x_m - q_m\right) + \left(q_m^T - x_m^T\right)A^{m,i}\left(x_i - q_i\right) \\
&+ \sum_{i\in V_L, j\in V_R}\sum_{E_C=(i,j)}\left(q_i^T - x_i^T\right)A^{i,j}\left(x_j - q_j\right) - \left(q_j^T - x_j^T\right)A^{j,i}\left(x_i - q_i\right) \\
&- \sum_{j\in V_R}\sum_{E_R=(j,m)}\left(q_j^T - x_j^T\right)A^{j,n}\left(x_n - q_n\right) + \left(q_n^T - x_n^T\right)A^{n,j}\left(x_j - q_j\right) \\
&= 0
\end{aligned}
$$

Before grouping the terms in the last step, we use the fact that when q_k is a fully mixed Nash equilibirum $(A^{i,k}q_k)_1 = (A^{i,k}q_k)_2 = \ldots = (A^{i,k}q_k)_{|S_i|}$, to make player i indifferent to playing any pure strategy. Hence, we have $\left(q_i^T - x_i^T\right)A^{i,k}q_k = 0$ for all players i. The last step is obtained after grouping the terms, to take the transpose of one of terms and apply the fact that $A^{i,k} = -A^{T^{k,i}}$ for all i, k, playing a bimatrix zero sum game. Similarly, we have $A^{i,k} = A^{T^{k,i}}$ for all coordination games played by i, k. ∎

We see that the **(z-z-z)** games (*e.g.*, $V_L = \{1, 2, 3\}$,$V_R = \phi$) and the **(c-c-z)** games (*e.g.*,

$V_L = \{2,3\}$,$V_R = \{1\}$) games are special cases of the zero-coordination-zero bipartite graph, whereas the **(c-c-c)** and the **(z-z-c)** games are not.

3.2 Z-Z-Z Game

As seen above, the **(z-z-z)** game is a special case of the zero-coordination-zero polymatrix game, which is composed of sums of KL divergences between the fully mixed Nash equilibirum and the trajectory of the system starting in the interior. Now, we leverage the Theorem 4 and see that the for the sum of the KL divergences to remain constant for any trajectory starting in the interior, the trajectories have to remain bounded away from the boundary, otherwise the term tends to infinity. In addition, it is shown in Theorem 3.3 of Piliouras and Shamma (2014), that, if the flow has an interior fixed point, then for each open set E that is bounded away from the boundary($bd\left(\times_i \Delta\left(S_i\right)\right)$) there exists orbits that intersect E infinitely often. The proof of which requires establishing a homeomorphism $g : \times_i int\left(\Delta\left(S_i\right)\right) \to \mathbb{R}^{\sum_i (|S_i|-1)}$. This establishes that the trajectories are volume preserving (when the flows start in the interior). For a special case with three players ($i = 3$) and two actions each $|S_i| = 2 \ \ \forall i$, the homeomorphism g is defined to be $\ln\left(\frac{x_{iR}}{x_{i0}}\right)$ and hence let us define $z_{iR} := \ln\left(\frac{x_{iR}}{x_{i0}}\right)$

Theorem 5. *A 3-player pairwise bimatrix symmetric* **(z-z-z)** *game, with trajectories that start in the interior of the strategy space, upon application of a homeomorphism* $z_{iR} := \ln\left(\frac{x_{iR}}{x_{i0}}\right)$*, are constrained to lie on a plane in* $\mathbb{R}^2$ *if the game has an interior Nash equilibirum. Otherwise the trajectories converge to the boundary. In addition, the trajectories are always periodic.*

Proof Firstly, we note that for a 3 player game with two strategies, we can rewrite the replicator dynamics as follows (with action 0 being shown here by default).

$$\ln\left(\frac{\dot{x_1}}{1-x_1}\right) = a_{12}x_2 + a_{13}x_3 + (c_{12} + c_{13})$$
$$\ln\left(\frac{\dot{x_2}}{1-x_2}\right) = -a_{12}x_1 + a_{23}x_3 + (c_{21} + c_{23})$$
$$\ln\left(\frac{\dot{x_3}}{1-x_3}\right) = -a_{13}x_1 - a_{23}x_2 + (c_{31} + c_{32})$$

where $a_{ij} = A^{i,j}_{1,1} + A^{i,j}_{1,1} - A^{i,j}_{1,2} - A^{i,j}_{2,1}$ for $i < j$, $c_{ij} = A^{i,j}_{1,2} - A^{i,j}_{2,2}$ and $c_{ji} = A^{T^{j,i}}_{1,2} - A^{T^{k,i}}_{2,2}$. The signs in the above equation are due to the fact that all the players are playing bimatrix zero sum games. Using the mapping defined above, we can make the following reduction assuming $a_{ij} \neq 0$ for all i, j (otherwise, the system automatically reduces to a dimension which is strictly less than 3). Eliminating the term with x_3 in the first two equations, we get

$$\begin{aligned} a_{13}\dot{z}_2 - a_{23}\dot{z}_1 &= a_{12}\big(-a_{13}x_1 - a_{23}x_2\big) + a_{13}(c_{21} + c_{23}) - a_{23}(c_{12} + c_{13}) \\ &= a_{12}\dot{z}_3 + a_{13}(c_{21} + c_{23}) - a_{23}(c_{12} + c_{13}) - a_{12}(c_{31} + c_{32}) \end{aligned}$$

Thus,

$$a_{13}\dot{z}_2 - a_{12}\dot{z}_3 - a_{23}\dot{z}_1 = K$$

where,

$$K = a_{13}\left(c_{21} + c_{23}\right) - a_{23}\left(c_{12} + c_{13}\right) - a_{12}\left(c_{31} + c_{32}\right)$$

Finally, integrating with respect to time from $t = 0$ to any generic point in time t, we get the following equation:

$$a_{13}z_2 - a_{23}z_1 - a_{12}z_3 = Kt + C_0 \tag{3}$$

If $K > 0$, the right hand side of the above equation tends to infinite as $t \to \infty$ and this implies the magnitude of atleast one of the z_i must approach infinity, which implies $x_i \to 0$ or $x_i \to 1$. This implies that the dynamics converge to the boundary. A similar argument follows for $K < 0$. Now when $K = 0$, the above equation is the plane given by:

$$a_{13}z_2 - a_{23}z_1 - a_{12}z_3 = C_0 \tag{4}$$

■

Hence, any trajectory starting in the interior is forced to lie in this plane. Through Poincare-Bendixon theorem, we know that the only possible limiting trajectories are limit cycles, period orbits or convergence to fixed points. However, Piliouras and Shamma (2014) also showed that the general network zero-sum game exhibits Poincare recurrence and hence the only possible limiting behaviour in this case is periodicity, such that the initial state is visited infinitely often.

Corollary 6. *The trajectories associated with the 3-player pairwise bimatrix symmetric* **(z-z-z)** *game, accommodates periodic orbits in the interior of the state space if and only if there is a Nash equilibrium in the interior of the state space.*

Proof The fully mixed Nash equilibrium of the system, correspond to the equilbria of the replicator dynamics which lie in the interior. These are characterized by the system of linear equations as follows:

$$\left(a_{12}\right) x_2 + \left(a_{13}\right) x_3 = -(c_{12} + c_{13}) \tag{5}$$

$$\left(-a_{12}\right) x_1 + \left(a_{23}\right) x_3 = -(c_{21} + c_{23}) \tag{6}$$

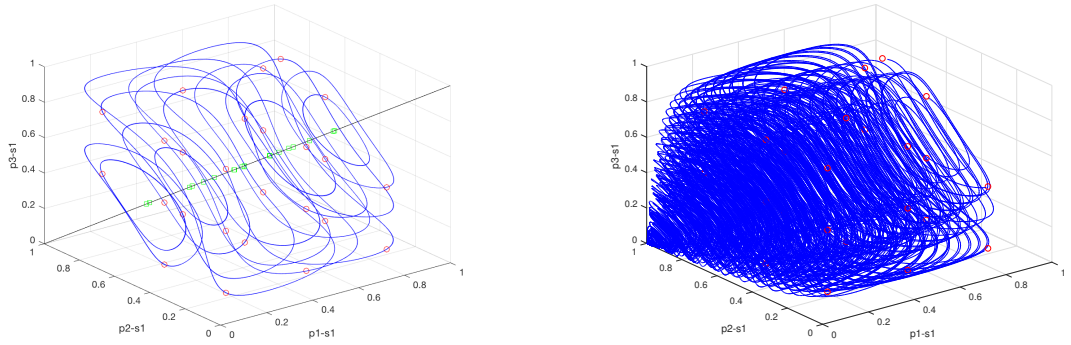
$$\left(-a_{13}\right) x_1 + \left(-a_{23}\right) x_2 = -(c_{31} + c_{32}) \tag{7}$$

Eliminating x_3 from the first two equations we get the following equation:

$$-a_{12}a_{13}x_1 - a_{12}a_{23}x_2 = (c_{21} + c_{23})a_{13} - (c_{12} + c_{13})a_{23} \tag{8}$$

■

Adding the above equation to $(-a_{12})$ times $\left(-a_{13}\right) x_2 + \left(-a_{23}\right) x_3 = -(c_{31} + c_{32})$, we get the left hand side of this to be 0 and the right hand side boils down to $K = a_{13}\left(c_{21} + c_{23}\right) - a_{23}\left(c_{12} + c_{13}\right) - a_{12}\left(c_{31} + c_{32}\right)$, as defined previously. Hence to have a continuum of equilibria it is necessary and sufficient that $K = 0$, and the system of linear equations will then have infinite solutions. From Theorem 5 we have that the limiting behavior is periodic in the interior of the state space. However, we also know from Theorem 5, when $K \neq 0$, or equivalently when this system has no interior equilibria, the limiting behavior is forced to go to the boundary of the state space thus proving that existence of interior periodic orbits implies the existence of a Nash equilibirum in the interior of the state space.



(a) $MP^Z_{1,2}(1,-1,-1,1), MP^Z_{1,3}(1,-1,-1,1), MP^Z_{2,3}(1,-1,-1,1)$ (b) $MP^Z_{1,2}(1,-1,-1,1), MP^Z_{1,3}(1,-1,-1,1), MP^Z_{2,3}(1.1,-1,-1,1$

Figure 2: Plots of trajectories in the **(z-z-z)** game.

3.3 C-C-Z Game

The **(c-c-z)** game is also a special case of the zero-coordination-zero bipartite game, with the invariant function being $H = -\left(\sum_{i=1}^{2}\sum_{k=0}^{1} q_{ik}\ln(x_{ik}) - \sum_{k=0}^{1} q_{3k}\ln(x_{3k})\right)$. Now, if we leverage the Theorem 4, we learn that this function can still remain constant when one of the terms go to ∞ and another term goes to $-\infty$ at the same rate. Hence it is impossible to conclude if the trajectories will be bounded away from the boundary (unlike **(z-z-z)** games). But we use homeomorphism $g : \times_i int(\Delta(S_i)) \to \mathbb{R}^{\sum_i(|S_i|-1)}$, that was defined previously and this is independent of the type of the game that is being played and is true due to the property of the replicator equations in general. We define $z_{iR} := \ln\left(\frac{x_{iR}}{x_{i0}}\right)$, similar to the **(z-z-z)** game.

Theorem 7. *A 3-player pairwise bimatrix symmetric* **(c-c-z)** *game, with trajectories that start in the interior of the strategy space, upon application of a homeomorphism* $z_{iR} := \ln\left(\frac{x_{iR}}{x_{i0}}\right)$*, are constrained to lie on a plane in* $\mathbb{R}^2$ *if the game has an interior Nash equilibirum. Otherwise the trajectories converge to the boundary. In addition, the trajectories can be periodic in the interior of the state space or exhibit convergence to limit cycles or fixed points.*

Proof Firstly, we note that for a 3 player game with two strategies, we can rewrite the replicator dynamics as follows (with action 0 being shown here by default).

$$\ln\left(\frac{\dot{x_1}}{1-x_1}\right) = a_{12}x_2 + a_{13}x_3 + (c_{12}+c_{13})$$
$$\ln\left(\frac{\dot{x_2}}{1-x_2}\right) = a_{12}x_1 + a_{23}x_3 + (c_{21}+c_{23})$$
$$\ln\left(\frac{\dot{x_3}}{1-x_3}\right) = a_{13}x_1 - a_{23}x_2 + (c_{31}+c_{32})$$

where $a_{ij} = A^{i,j}_{1,1} + A^{i,j}_{1,1} - A^{i,j}_{1,2} - A^{i,j}_{2,1}$ for $i<j$, $c_{ij} = A^{i,j}_{1,2} - A^{i,j}_{2,2}$ and $c_{ji} = A^{T^{j,i}}_{1,2} - A^{T^{k,i}}_{2,2}$. Using the mapping defined above, we can do the following (assuming $a_{ij} \neq 0$ for all i,j, with similar reasoning provided in Theorem 5). Eliminating the term with x_3 in the first

two equations, we get

$$\begin{aligned} a_{13}\dot{z}_2 - a_{23}\dot{z}_1 &= a_{12}\big(a_{13}x_1 - a_{23}x_2\big) + a_{13}(c_{21}+c_{23}) - a_{23}(c_{12}+c_{13}) \\ &= a_{12}\dot{z}_3 + a_{13}(c_{21}+c_{23}) - a_{23}(c_{12}+c_{13}) - a_{12}(c_{31}+c_{32}) \end{aligned}$$

Thus,

$$a_{13}\dot{z}_2 - a_{12}\dot{z}_3 - a_{23}\dot{z}_1 = K$$

where,

$$K = a_{13}\,(c_{21}+c_{23}) - a_{23}\,(c_{12}+c_{13}) - a_{12}\,(c_{31}+c_{32})$$

Finally, integrating with respect to time from $t = 0$ to any generic point in time t, we get the following equation:

$$a_{13}z_2 - a_{23}z_1 - a_{12}z_3 = Kt + C_0 \tag{9}$$

If $K > 0$, the right hand side of the above equation tends to infinite as $t \to \infty$ and this implies the magnitude of at least one of the z_i must approach infinity, which implies $x_i \to 0$ or $x_i \to 1$. This implies that the dynamics converge to the boundary. A similar argument follows for $K < 0$. Now when $K = 0$, the above equation is the plane given by:

$$a_{13}z_2 - a_{23}z_1 - a_{12}z_3 = C_0 \tag{10}$$

■

Hence, any trajectory starting in the interior is forced to lie in this plane. Through Poincare-Bendixon theorem, we know that the only possible limiting trajectories are limit cycles, period orbits or convergence to fixed points.

Corollary 8. *If the trajectories associated with the 3-player pairwise bimatrix symmetric* **(c-c-z)** *game accommodates periodic orbits in the interior of the state space then there exists a Nash equilibrium in the interior of the state space.*

Proof The fully mixed Nash equilibrium of the system, correspond to the equilbria of the replicator dynamics which lie in the interior. These are characterized by the system of linear equations as follows:

$$(a_{12})\,x_2 + (a_{13})\,x_3 = -(c_{12}+c_{13}) \tag{11}$$

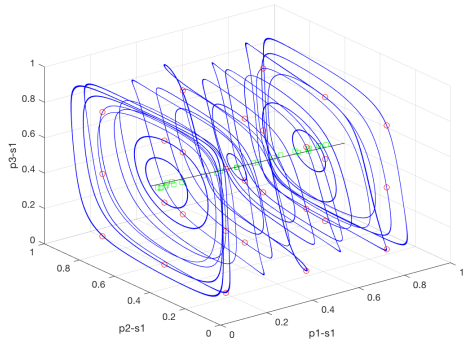
$$(a_{12})\,x_1 + (a_{23})\,x_3 = -(c_{21}+c_{23}) \tag{12}$$

$$(a_{13})\,x_1 + (-a_{23})\,x_2 = -(c_{31}+c_{32}) \tag{13}$$

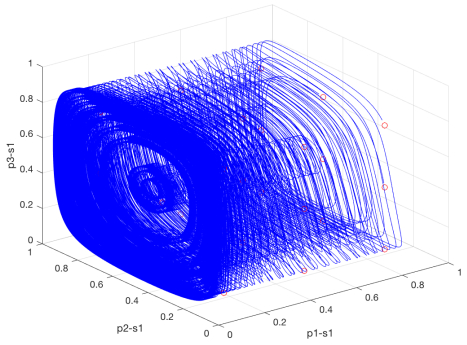
Eliminating x_3 from the first two equations we get the following equation:

$$a_{12}a_{13}x_1 - a_{12}a_{23}x_2 = (c_{21}+c_{23})a_{13} - (c_{12}+c_{13})a_{23} \tag{14}$$

Adding the above equation to $(-a_{12})$ times $(a_{13})\,x_2 + (-a_{23})\,x_3 = -(c_{31}+c_{32})$, we get the left hand side of this to be 0 and the right hand side boils down to $K = a_{13}\,(c_{21}+c_{23}) - a_{23}\,(c_{12}+c_{13}) - a_{12}\,(c_{31}+c_{32})$, as defined previously. In addition, from the previous theorem we see that when $K \neq 0$ the limiting behavior is forced to go to the boundary of the state space. Hence if the there is a periodic orbit in the interior then K should be equal to 0. But as seen above, if $K = 0$ the system of linear equations will then have infinite solutions which implies the existence of a Nash equilibirum in the interior of the state space. ■

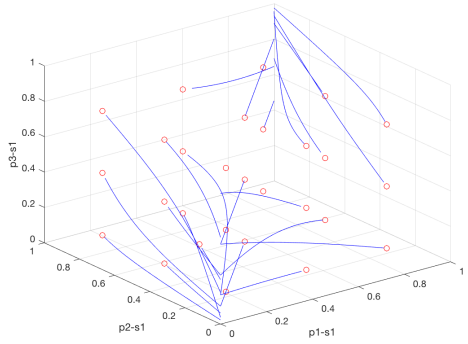


(a) $MP^{C}_{1,2}(1,-1,-1,1)$,$MP^{C}_{1,3}(1,-1,-1,1)$,5∗ $MP^{Z}_{2,3}(1,-1,-1,1)$



(b) $MP^{C}_{1,2}(1,-1,-1,1)$,$MP^{C}_{1,3}(1,-1,-1,1)$,5∗ $MP^{Z}_{2,3}(1.1,-1,-1,1)$

Figure 3: Plots of trajectories in the **(c-c-z)** game, with a normal and perturbed payoff on the zero-sum edge.



(a) $MP^{C}_{1,2}(1,-1,-1,1)$,$MP^{C}_{1,3}(1,-1,-1,1)$,$MP^{Z}_{2,3}(1,-1,-1,1)$

Figure 4: Plots of trajectories in the **(c-c-z)** game, converging to fixed points on the boundary.

3.4 C-C-C Game

The fully coordinated game, is not a special case of the more general zero-coordination-zero bipartite game and hence we have to resort to other methods to perform the global analysis in this case. One such result is proved in Kleinberg et al. (2009) for congestion games. We look at the social welfare of this system and prove that it serves as a Lyapunov (Potential) function for the system.

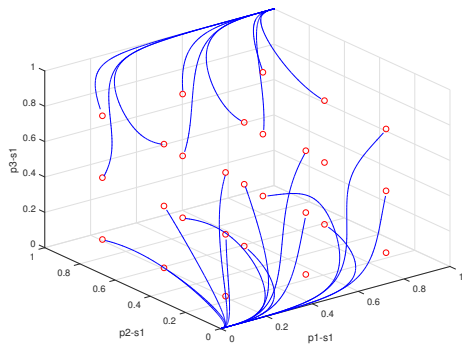
Theorem 9. *The social welfare of the system* $SW = \sum_i u_i$ *is a Lyapunov function. The dynamics converge to equilibria.*

Proof Let us consider the derivative of SW

$$\begin{aligned}
S\dot{W} &= \sum_i \sum_{R \in S_i} \frac{\partial SW}{\partial x_{iR}} \dot{x}_{iR} \\
&= \sum_i \sum_{R \in S_i} u_i(R) x_{iR} \left(u_i(R) - \sum_{R'} u_i(R') x_{iR'} \right) \\
&= \sum_i \sum_{R \in S_i} u_i(R) x_{iR} \left(\sum_{R'} u_i(R) x_{iR'} - \sum_{R'} u_i(R') x_{iR'} \right) \\
&= \sum_i \sum_{R \in S_i} x_{iR} \sum_{R'} x_{iR'} \left(u_i^2(R) - u_i(R) u_i(R') \right) \\
&= \sum_i \sum_{R \in S_i} \sum_{R' \in S_i} x_{iR} x_{iR'} \left(u_i^2(R) - u_i(R) u_i(R') \right) \\
&= \sum_i \sum_{R < R' \in S_i} x_{iR} x_{iR'} \left(u_i^2(R) + u_i^2(R') - 2 u_i(R) u_i(R') \right) \\
&= \sum_i \sum_{R < R' \in S_i} x_{iR} x_{iR'} \left(u_i(R) - u_i(R') \right)^2
\end{aligned}$$

The last step is making use of the fact that each player is playing a coordination game and hence the utilities are the same for the same strategy, and hence they can be grouped into the square terms. This means that the social welfare of the system always increases with time and hence serves as a Lyapunov function. This easily implies that the system converges to an equilibrium. ∎

By Kleinberg et al. (2009) it is known that in potential games generically only pure Nash equilibria are stable and thus the system typically converges to a pure Nash equilibrium as seen in the experiments.



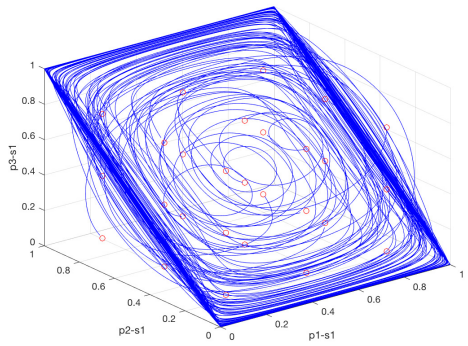
(a) $MP^C_{1,2}(1,-1,-1,1)$,$MP^C_{1,3}(1,-1,-1,1)$,5∗ $MP^C_{2,3}(1,-1,-1,1)$

Figure 5: Plots of trajectories in **(c-c-c)** game.

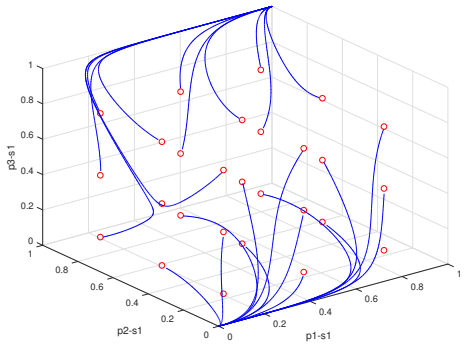
3.5 Z-Z-C Game

The z-z-c game is not a special case of any of the above games seen before and as such no reduction is possible, akin to the **(z-z-z)** and the **(c-c-z)** games. Moreover,

the behavior of this system is more complicated than the pure coordination game and does not always converge to equilibria. This is expected due to the presence of zero-sum edges. We present some simulation results for the **(z-z-c)** game below and try to study the dynamics experimentally.



(a) $MP^Z_{1,2}(1,-1,-1,1)$,$MP^Z_{1,3}(1,-1,-1,1)$,0.2∗ $MP^C_{2,3}(1,-1,-1,1)$



(b) $MP^Z_{1,2}(1,-1,-1,1)$,$MP^Z_{1,3}(1,-1,-1,1)$,5∗ $MP^C_{2,3}(1,-1,-1,1)$

Figure 6: Plots of trajectories in the **(z-z-c)** game, showing cases with convergence to fixed points and limit cycle.

4 Discussion of Results

We discuss some of the simulation results in detail here.

4.1 z-z-z games

The fully zero-sum games in figure 2 exhibit periodic orbits in the interior for different interior starting points (shown as red points). Moreover, the time average of the system trajectories (shown as green markers), that lie completely in the interior coincide with line of equilibria (shown in black). This is a useful characterization of the average behavior of the system in terms of the Nash equilibria. The exception to the above rule is the case when one of the zero-sum games is $MP^Z_{2,3}(1.1,-1,-1,1)$ and this slight perturbation causes the system to converge to the boundary at an exponential rate, as we had shown while deriving the reduction for the **(z-z-z)** game. Due to the payoff asymmetry we have, $K \neq 0$ and hence there are no interior Nash equilibria.

4.2 c-c-z games

The plots in the **(c-c-z)** game were obtained by keeping the coordination games the same for all the cases and varying the extent of the zero-sum game by changing its payoffs. In figure 3, we observe that with the particular set of game parameters, the system exhibits behavior that is very much similar to a fully zero-sum game, including the fact that it is not robust to perturbations in the payoff matrix, which causes the system to converge to the boundary. However, by increasing the payoffs in the coordination game, we see that the system converges to the vertices (just like in a full coordination game).

4.3 c-c-c games

The results of the **(c-c-c)** game are in line with the fact that they are potential games and hence the Lyapunov function which is the social welfare of the system increases along the trajectory of the system. Convergence to the boundary (vertices), can be seen in figure 5. A similar behavior was observed for other changes in the parameters of one of the coordination game payoffs.

4.4 z-z-c games

Finally, the case of **(z-z-c)** we show experimentally that it can exhibit limit behavior that is not observed in any of other settings including boundary limit cycles where the two agents who are coordinating and the single agent who is playing two zero-sum games, take turns best responding simultaneously to each other. In this case, x_1 gets arbitrarily close to 0 or 1 and has to cycle because it is playing a zero-sum game against two coordinating players, hence, it can be seen that x_2 and x_3 move along $x_2 = x_3$ line. Furthermore, increasing the strength of the coordination game, results in the breaking of this limit cycle and then attains convergence.

4.5 Related Work

The fact that equilibria do not suffice to understand the behavior of replicator dynamics even in simple games dates at least back to the work of Eshel and Akin Eshel and Akin (1983) where they established the instability of mixed Nash equilibria for replicator dynamics in (single-population) games; for a detailed discussion, see Hofbauer and Sigmund (1998). The existence of constant of motions for the replicator dynamics allowed the characterization of certain classes of (two-player symmetric random-matching) games as *Hamiltonian* systems (i.e. as dynamics possessing a Hamiltonian function that foliates the space of population states into invariant manifolds) Hofbauer (1996). Piliouras and Shamma Piliouras and Shamma (2014) were the first to establish Poincaré recurrence for (network) zero-sum games in the context of replicator dynamics. These results have recently been generalized to more expanded classes of games Piliouras et al. (2014) and dynamics including follow-the-regularized leader dynamics Mertikopoulos et al. (2017). Finally, current work extends such Poincaré recurrence results even in the case of time-evolving Rock-Paper-Scissors games where the exact parameters of the game are a function of the state of the game Mai et al. (2017). None of these results can establish periodicity, which is a much stronger recurrence property and requires specialized arguments. Our focus on this paper is on continuous-time systems. The dynamics of discrete time multi-agent systems (e.g. Multiplicative Weights Dynamics) in contrast can be chaotic even in simple symmetric two by two games Palaiopanos et al. (2017). The role of relationship triangles, such as "the friend of my friend is my friend" and "the enemy of my friend is my enemy" has been studied in evolutionary models of structural changes in networks by Jon Kleinberg and colleagues Marvel et al. (2011). More recently, the evolution of beliefs has been studied in such signed networks where a plus sign corresponds to friends and a minus sign corresponds to enemies Shi et al. (2016). The evolution of selfish behavior within such triangles has not been studied before as far as we know.

5 Conclusion

In this paper, we analyze the behavior of replicator dynamics, arguably the most well known evolutionary dynamics, in three player polymatrix (triangle) games where each edge game is either a two-by-two zero sum or coordination game. These settings encode triadic friend-or-foe interactions. Relationship triangles Marvel et al. (2011) is a basic building block of social networks and are thus a natural setting on which to study multi-agent learning behavior. From the perspective of dynamical systems since these systems are nonlinear and have three independent variables chaotic behavior is possible. Nevertheless, no such behavior is observed with the systems either converging to equilibria, to limit cycles or are periodic. Given that four dimensional replicator systems can exhibit chaos Skyrms (1992) triangle friend-or-foe games are shown to lie on a sweet spot between complexity and simplicity. They allow for complex but predictable interactions. This might be an indicator of the practical importance of relationship triangles for social networks as they allow for richer social dynamics that are still manageable and interpretable.

Acknowledgements

Sai Ganesh Nagarajan acknowledges the SUTD Presidential Graduate Fellowship. Georgios Piliouras acknowledges SUTD grant SRG ESD 2015 097, MOE AcRF Tier 2 Grant 2016-T2-1-170, NRF grant NRF2016NCR-NCR002-028 and NRF fellowship NRF-NRFF2018-07.

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