Optimization Despite Chaos: Convex Relaxations to Complex Limit Sets via Poincaré Recurrence

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Abstract

It is well understood that decentralized systems can, through network interactions, give rise to complex behavior patterns that do not reflect their equilibrium properties. The challenge of any analytic investigation is to identify and characterize persistent properties despite the inherent irregularities of such systems and to do so efficiently. We develop a novel framework to address this challenge.

Our setting focuses on evolutionary dynamics in network extensions of zero-sum games. Such dynamics have been shown analytically to exhibit chaotic behavior which traditionally has been thought of as an overwhelming obstacle to algorithmic inquiry. We circumvent these issues as follows: First, we combine ideas from dynamical systems and game theory to produce topological characterizations of system trajectories. Trajectories capture the time evolution of the system given an initial starting state. They are complex, and do not necessarily converge to limit points or even limit cycles. We provide tractable approximations of such limit sets. These relaxed descriptions involve simplices, and can be computed in polynomial time. Next, we apply standard optimization techniques to compute extremal values of system features (*e.g.* expected utility of an agent) within these relaxations. Finally, we use information theoretic conservation laws along with Poincaré recurrence theory to argue about tightness and optimality of our relaxation techniques.

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1 Introduction

Complex networks are increasingly integrated in the very fabric of our society. As much as these networks bring people together and facilitate cooperation, they also give rise to unexpected behavior patterns. Over the last decade, the prevalence of such issues has risen dramatically following a number of paradigm-shifting events such as the meteoric rise of the Internet as a social networking tool, the painful realization of the extent of inter-connectivity of the global economy as well as the necessity of international cooperation for addressing global sustainability concerns.

Within computer science, algorithmic game theory has been developed in an effort to provide a quantitative lens for studying such systems. With classic game theory as a guiding beacon, Nash equilibrium analysis quickly became the de facto solution standard. Despite its prominent role, this direction has been the subject of much criticism. In games with multiple equilibria, it is unclear how agents are expected to coordinate on one. Furthermore, identifying a single Nash equilibrium may involve unreasonable expectations on agent communication and computation. Finally, natural adaptive play does not converge to Nash equilibria in general games. In fact, there exist games with constant number of agents and strategies in which natural learning dynamics converge to limit cycles of optimal social welfare which can be *arbitrarily better* than the social welfare of the *best equilibrum* [23, 25].

Equilibrium analysis has emerged pretty much unscathed from these remarks. To counter this disconnect between the proposed equilibrium methodology and the usually more intricate system behavior, proponents of the equilibrium approach rightfully point out that equilibrium analysis already raises significant analytical hurdles. Once we move past the safe haven of Nash equilibria, which capture (possibly unstable) fixed points of learning dynamics, the topological waters become deep and turbulent fast. Indeed, in [29] Sato *etal.* make this argument rather convincingly by showing that even simple zero-sum games (perturbed versions of Rock-Paper-Scissors) suffice to give rise to chaotic behavior under classic evolutionary dynamics. Specifically, they analytically verify high sensitivity of system trajectories to initial conditions and provide experimental evidence of dense interweaving between quasi-periodic tori and chaotic orbits.

A cursory glance at the rate of progress of theoretical work on disequilibrium dynamics is rather telling about the complexities involved. Soon after Nash formalized his ideas about equilibrium, Shapley [31] established that simple learning dynamics do not converge to equilibrium. The game in question is a two agent non-zero sum variant of Rock-Paper-Scissors. Followup work by Jordan[21] established analogous results for three agent games. These results have been revisited and reestablished via different routes [14]. Within computer science recent work about non-convergence of learning dynamics in games focuses on variants of these games and dynamics [8, 23, 3]. The unified characteristics of all these works, which are already present in the work of Shapley, are that the networks are of constant size, the dimension of the action space is constant, and that the disequilibrium behavior is as simple as possible. Specifically, the dynamics exhibit a single polygon-shaped cyclic attractor with a constant number of vertices. These attractors are usually referred to as Shapley polygons.

Another key common characteristic of this prior work is that it is largely qualitative in nature. The goal is to present a simple and easy to convey message. Given that chaotic phenomena can arise even in simple examples, researchers traverse a fine line of identifying classes of games whose hardness is exactly right. The settings must be simple enough so that the true system behavior can be expressed concisely and with perfect precision, and rich enough so that the resulting picture diverges significantly from that of standard equilibrium analysis.

Such approaches are undoubtedly useful in terms of cultivating a collective mindset about the importance of meeting these analytical challenges fully. However, if nonequilibrium analysis hopes to graduate to the level of a concrete and actionable scientific framework one needs to show how progress in this area can translate to novel algorithmic insights. How can one compute, or even efficiently encode, regularities in an ever transient complex environment? We hope to shed some light along this direction.

Our results We study an evolutionary class of dynamics, the replicator equation, in a network generalization of constant-sum games known as separable zero-sum multiplayer games. These are polymatrix graphical games where the sum of all agents' payoffs is always equal to zero. This setting incorporates both the complexity of chaotic trajectories as well as computational intricacies that arise from its combinatorial structure. Specifically, we present the first to our knowledge analysis of nonequilibrium dynamics for a class of games of arbitrary dimension (number of agents, number of strategies).

We introduce formal notions of persistent properties for nonconverging dynamics. Roughly speaking, a subspace of the state space defines a persistent property if for all initial conditions systems trajectories eventually move away from the complement of the subspace and never return to it. Arguing about persistent properties requires topological characterizations of system trajectories. In generic (network) zero-sum games no trajectory converges to equilibrium. We combine elements from theory of dynamical systems, game theory and online learning theory to show that the limit sets of all starting points lie within a specific subspace. This subspace is a product of simplices and expresses the set of all mixed strategy profiles whose support matches that of the Nash equilibrium of maximum support. Furthermore, we utilize an information theoretic conservation law to extend Poincaré recurrence theory to our setting and argue about the optimality of the relaxation.

In the second part of the paper we show how we can combine these topological characterizations with algorithmic tools to make tight predictions about the possible range of values of interesting system features. We start by extending ideas from algorithmic game theory to show that we can compute these simplex-based relaxed descriptions efficiently. Next, we show that solving optimization problems over such relaxed subsets suffices to track extremal recurrent value problems for continuous features of the state space (*e.g.* expected utility of an agent). Finally, we discuss some concrete applications of our framework towards identifying margins of survival as well as optimal performance measures for competing companies in networked economies.

A benchmark example: Rock-Paper-Scissors Let's consider the benchmark case of the Rock-Paper Scissors game in order to convey some concrete insights. A key observation is that starting from any (interior) starting point the sum of the K-L divergences between the unique fully mixed Nash equilibrium strategy of each agent and her evolving strategy remains constant as the system moves forward in time. A useful fact to keep in mind is that K-L divergence can be thought of as a (pseudo)-metric.[1] The fact that this pseudo-distance between the Nash equilibrium and the evolving system state remains constant is useful. Intuitively, it implies that the replicator dynamics starting from a fully mixed state will: a) not converge to the Nash equilibrium (K-L divergence cannot drop to zero); b) will not come arbitrarily close to the boundary (K-L divergence cannot blow up to infinity). In fact, for any two initial conditions (mixed strategy profiles) x and y with distinct initial K-L divergences from the Nash equilibrium the corresponding limit behavior must be distinct. This richness in limit behavior (with infinitely many different possible limit behaviors) comes at a contrast with the established intuition about zero-sum games having a unique behavioral "solution".

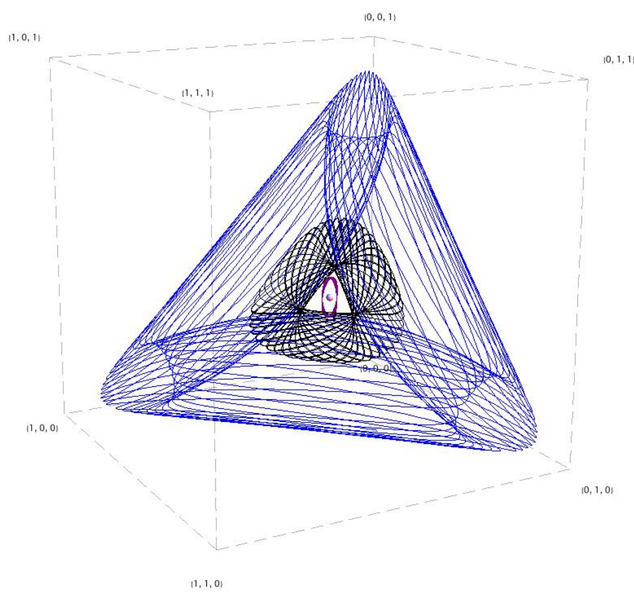

Figure 1: Sample orbits of replicator dynamics in Rock-Paper-Scissors (row agent's mixed strategy)

More to the point, our analysis implies that given any open set of initial conditions that is bounded away from the boundary there exist replicator trajectories that revisit this set infinitely often. Before we examine how one could prove such a statement, let's examine why such a statement is useful. Let's imagine an outside observer that makes a measurement of a continuous observable feature[2] every time this trajectory "hits" the open set. If we define the open set as an (open) ball of radius $\delta > 0$, then for any $\epsilon > 0$ we can choose such a $\delta > 0$ such that any two such measurements are within an ϵ of each other. For all practical purposes these observations become indistinguishable for small enough ϵ, δ. Since these (range of) values can be revisited infinitely often (for properly chosen initial conditions) a predictive statement that holds regardless of initial conditions must account for them. Using properly chosen open balls of initial conditions we can approximate any value of a continuous feature (anywhere on the state space) within an error of ϵ for any $\epsilon > 0$. Therefore, without explicit knowledge of initial conditions, the only closed set of predictions that we can make for any continuous feature is the trivial one, *i.e.*, the set that includes all possible range of values. So, not only is the unique Nash equilibrium not predictive of the actual day-to-day agent behavior (or any continuous observable feature of it) but from an optimization point-of-view no nontrivial prediction can be made!

This recurrence of observed values would be immediately true if all trajectories where either fixed points (*i.e.* Nash equilibria) or formed limit cycles, that is, if they looped perfectly into themselves. Unfortunately, this concise characterization is not true even for separable zero-sum games with a unique fully mixed Nash equilibrium[3]. We need to apply a more relaxed and powerful notion of recurrence that dates back to Poincaré and his celebrated 1890's memoir on the three body problem [28]. Poincaré's recurrence theorem states that the trajectories of certain systems return arbitrarily close to their initial position and that they do this an infinite number of times. Combining the fact that in Rock-Paper-Scissors given any initial fully mixed strategy the system state stays away from boundary along with other technical properties of the replicator dynamic, we show how to extend the implications of this theorem to our setting and prove recurrence of open sets. Specifically, we establish topological conjugacies, which is a powerful notion of equivalence for flows, between replicator flows and specific classes of conservative systems for which Poincaré's recurrence theorem is known to apply.

Related Work Persistent cyclic or chaotic patterns are a ubiquitous reality of natural systems. Research on the intersection of mathematical ecology and dynamical systems dating back to the 70's[10, 11, 13, 16] has developed a wide range of tools for addressing the following question: Given a dynamical system capturing

[1] The K-L divergence between two distributions is non-negative and it is equal zero if and only if the two distributions match. It is also finite for any two distributions with the same support.

[2] For now, think of a feature as a function from the state space to the real line.

[3] Such an example is a setting consisting of two independent copies of matching pennies where the utilities in one of the games have been scaled up by an irrational number.

the interaction of different species in a population is it true that all species will survive in the long run? In our work we apply these tools towards understanding other abstract properties of our state space. More extended surveys of results in the area can be found here[20, 18].

Our work was inspired by recent insights on network generalizations of zero-sum games[9, 6]. Their approach is mostly focused on the properties of Nash equilibria. Nash equilibrium computation is shown to be tractable. Furthermore, the set of Nash equilibria is convex. Lastly, the *time-average* of no-regret dynamics converges weakly to the set of Nash equilibria. On the contrary our work focuses on the "day-to-day" properties of the trajectories of the replicator dynamic, an explicit regret-minimizing dynamic.

In terms of analyzing replicator dynamics in settings of interest, the work of Kleinberg, Piliouras and Tardos[24] show how replicator when applied to generic potential games leads to the collapse of the randomization of an initial generic state. The story we are presenting here is essentially the dual of that in potential games. Replicator dynamics leads to the preservation of as much of the initial randomness of the system as possible. Our tools for analyzing replicator dynamics build upon the work of Akin and Losert[1] on symmetric zero-sum games without however following in the footsteps of their symplectic forms approach. The other major influence to our work has come from the work of Hofbauer, who amongst other contributions in the area was the first to offer formal connections between replicator dynamics and conservative systems via smooth invertible mappings[17, 18].

2 Preliminaries

In this section, we introduce the necessary concepts and notions that will enable us to express our points formally. Several of the concepts such as the replicator dynamic and the overview of dynamical systems theory are classic, however, other ones such as that of feature, property and persistent property are novel.

2.1 Separable zero-sum multiplayer game A graphical polymatrix game is defined via an undirected graph $G = (V, E)$, where V corresponds to the set of agents of the game and where every edge corresponds to a bimatrix game between its two endpoints/agents. We denote by S_i the set of strategies of agent i. We denote the bimatrix game on edge $(i, k) \in E$ via a pair of payoff matrices: $A^{i,k}$ of dimension $|S_i| \times |S_k|$ and $A^{k,i}$ of dimension $|S_k| \times |S_i|$. Let $s \in \times_i S_i$ be a strategy profile of the game. We denote by $s_i \in S_i$ the respective strategy of agent i. Similarly, we denote by $s_{-i} \in \times_{j \in V \setminus i} S_j$ the strategies of the other agents. The payoff of agent $i \in V$ in strategy profile s is equal to the sum of the payoffs that agent i receives from all the bimatrix games she participates in. Specifically, $u_i(s) = \sum_{(i,k)\in E} A^{i,k}_{s_i,s_k}$.

A randomized strategy x for agent i lies on the simplex $\Delta(S_i) = \{p \in \mathbb{R}^{|S_i|}_+ : \sum_i p_i = 1\}$. A randomized strategy x is said to be fully mixed if it lies in the interior of the simplex, *i.e.* if $x_i > 0$ for all strategies $i \in S_i$. Payoff functions are extended to randomized strategies in the usual multilinear fashion. A (mixed) Nash equilibrium is a profile of mixed strategies such that no agent can improve her (expected) payoff by unilaterally deviating to another strategy.

Definition 2.1. *(Separable zero-sum multiplayer games) [6] A separable zero-sum multiplayer game GG is a graphical polymatrix game in which, for any pure strategy profile, the sum of all players' payoffs is zero. Formally,* $\forall s \in \times_i S_i, \sum_i u_i(s) = 0$.

Zero-sum games trivially have this property. Any graphical games in which every edge is a zero-sum game also belongs to this class. These games are referred to as pairwise zero-sum polymatrix games [9]. If the edges are allowed to be arbitrary constant-sum games then the corresponding games are called pairwise constant-sum polymatrix games. There exists [6] a (polynomial-time computable) payoff preserving transformation from every separable zero-sum multiplayer game to a pairwise constant-sum polymatrix game (*i.e.*, a game played on a graph with agents on the nodes and two-agent games on each edge such for each $i, k \in V : A^{k,i} = c_{\{k,i\}}\mathbf{1} - A^{\mathrm{T}}_{i,k}$ and $\mathbf{1}$ the all-one matrix). We will use this representation in the rest of the paper.

2.2 Replicator Dynamics The replicator equation is among the basic tools in mathematical ecology, genetics, and mathematical theory of selection and evolution. In its classic continuous form, it is described by the following differential equation:

$$\dot{x}_i \triangleq \frac{dx_i(t)}{dt} = x_i[u_i(x) - \hat{u}(x)], \quad \hat{u}(x) = \sum_{i=1}^{n} x_i u_i(x)$$

where x_i is the proportion of type i in the population, $x = (x_1, \ldots, x_m)$ is the vector of the distribution of types in the population, $u_i(x)$ is the fitness of type i, and $\hat{u}(x)$ is the average population fitness. The state vector x can also be interpreted as a randomized strategy of an adaptive agent that learns to optimize over its m possible actions given an online stream of payoff vectors. The right hand-size of the replicator equation defines a continuously differentiable function defined on (the interior) of the associated simplex. This function is referred to as the vector field ξ. Since the replicator allows to associate stream of payoff vectors to mixed strategies, it can be employed in any distributed optimization setting by having each agent update her strategy according to it. A fixed point of the replicator is a point where its vector field is equal to the zero vector. Such points are stationary, *i.e.*, if the system starts off from such a point it stays there. An interior point of the state space is a fixed point for the replicator if and only if it is a fully mixed Nash equilibrium of the game. The interior (the boundary) of the state space $\times_i \Delta(S_i)$ are invariants for the replicator. We typically analyze the behavior of the replicator from a generic interior point, since points of the boundary can be captured as interior points of lower dimensional systems. Summing all this up, our model is captured by the following system:

$$\dot{x}_{iR} = x_{iR}\big(u^i(R) - \sum_{R' \in S_i} x_{iR'} u^i(R')\big)$$

for each agent $i \in N$, action $R \in S_i$, and where we define $u^i(R) = \mathrm{E}_{s_{-i} \sim x_{-i}} u_i(R, s_{-i})$.

The replicator dynamic enjoys numerous desirable properties such as universal consistency (no-regret)[12, 19], connections to the principle of minimum discrimination information (Occam's rajor, Bayesian updating), disequilibrium thermodynamics[22, 30], classic models of ecological growth (e.g. Lotka-Volterra equations[18]), as well as several well studied discrete time learning algorithms (e.g. Multiplicative Weights algorithm[24, 2]).

2.3 Topology of dynamical systems In an effort to make our work as standalone as possible we provide a quick introduction to the main ideas in the topology of dynamical systems. Our treatment follows that of [32], the standard text in evolutionary game theory, which itself borrows material from the classic book by Bhatia and Szegö[5].

Since our state space is compact and the replicator vector field is Lipschitz-continuous, we can present the unique solution of our ordinary differential equation as a continuous map $\Phi : \mathcal{S} \times \mathbb{R} \to \mathcal{S}$ called flow of the system. Fixing starting point $x \in \mathcal{S}$ defines a function of time which captures the trajectory (orbit, solution path) of the system with the given starting point. This corresponds to the graph of $\Phi(x, \cdot) : \mathbb{R} \to \mathcal{S}$, *i.e.*, the set $\{(t, y) : y = \Phi(x, t) \text{ for some } t \in \mathbb{R}\}$. In simple terms, the trajectory captures the evolution of the state of a system given an initial starting point. On the other hand, by fixing time t, we obtain a map of the state space to itself $\Phi^t : \mathcal{S} \to \mathcal{S}$. The resulting family of mappings exhibits the standard group properties such as identity (Φ^0) and existence of inverse (Φ^{-t}), and closure under composition $\Phi^{t_1} \circ \Phi^{t_2} = \Phi^{t_1+t_2}$.

If the starting point x does not correspond to an equilibrium, then we wish to capture the asymptotic behavior of the system (informally the limit of $\Phi(x, t)$ when t goes to infinity). Typically, however, such functions do not exhibit a unique limit point so instead we study the set of limits of all possible convergent subsequences. Formally, given a dynamical system $(\mathbb{R}, \mathcal{S}, \Phi)$ with flow $\Phi : \mathcal{S} \times \mathbb{R} \to \mathcal{S}$ and a starting point $x \in \mathcal{S}$, we call point $y \in \mathcal{S}$ an ω-limit point of the orbit through x if there exists a sequence $(t_n)_{n \in \mathbb{N}} \in \mathbb{R}$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \Phi(x, t_n) = y$. Alternatively the ω-limit set can be defined as: $\omega_\Phi(x) = \cap_t \overline{\cup_{\tau \geq t} \Phi(x, \tau)}$.

Finally, the boundary of a subset S is the set of points in the closure of S, not belonging to the interior of S. Generally, given a subset S of a metric space with metric $dist$, then x is an interior point of S if there exists $r > 0$, such that y is in S whenever the $dist(x, y) < r$. In the typical case of a Euclidean space, then simply x is an interior point if there exists an open set centered at it which is contained in S. An element of the boundary of S is called a boundary point of S. We denote the boundary of a set S as bd(S) and the interior of S as int(S). In the case of the replicator dynamics where the state space S corresponds to a product of agent (mixed) strategies we will denote by $\Phi_i(x, t)$ the projection of the state on the simplex of mixed strategies of agent i.

Liouville's Formula Liouville's formula can be applied to any system of autonomous differential equations with a continuously differentiable vector field ξ on an open domain of $\mathcal{S} \subset \mathbb{R}^k$. The divergence of ξ at $x \in \mathcal{S}$ is defined as the trace of the corresponding Jacobian at x, *i.e.*, $\mathrm{div}[\xi(x)] = \sum_{i=1}^k \frac{\partial \xi_i}{\partial x_i}(x)$. Since divergence is a continuous function we can compute its integral over measurable sets $A \subset \mathcal{S}$. Given any such set A, let $A(t) = \{\Phi(x_0, t) : x_0 \in A\}$ be the image of A under map Φ at time t. $A(t)$ is measurable and is volume is $\mathrm{vol}[A(t)] = \int_{A(t)} dx$. Liouville's formula states that the time derivative of the volume $A(t)$ exists and is equal to the integral of the divergence over $A(t)$:

$$\frac{d}{dt}[A(t)] = \int_{A(t)} \mathrm{div}[\xi(x)] dx.$$

A vector field is called divergence free if its divergence is zero everywhere. Liouville's formula trivially implies that volume is preserved in such flows.

Poincaré's recurrence theorem The notion of recurrence that we will be using in this paper goes back to Poincaré and specifically to his study of the three-body problem. In 1890, in his celebrated work[28], he proved that whenever a dynamical system preserves volume almost all trajectories return arbitrarily close to their initial position, and they do so an infinite number of times. More precisely, Poincaré established the following:

THEOREM 2.1. *[28, 4] If a flow preserves volume and has only bounded orbits then for each open set there exist orbits that intersect the set infinitely often.*

Homeomorphisms, Diffeomorphisms and Conjugacy of Flows A function f between two topological spaces is called a *homeomorphism* if it has the following properties: f is a bijection, f is continuous, and f has a continuous inverse. A function f between two topological spaces is called a *diffeomorphism* if it has the following properties: f is a bijection, f is continuously differentiable, and f has a continuously differentiable inverse.

DEFINITION 2.2. *(Topological conjugacy) Two flows $\Phi^t : A \to A$ and $\Psi^t : B \to B$ are conjugate if there exists a homeomorphism $g : A \to B$ such that for each $x \in A$ and $t \in \mathbb{R}$:*

$$g(\Phi^t(x)) = \Psi^t(g(x)).$$

Furthermore, two flows $\Phi^t : A \to A$ and $\Psi^t : B \to B$ are *diffeomorhpic* if there exists a diffeomorphism $g : A \to B$ such that for each $x \in A$ and $t \in \mathbb{R}$ $g(\Phi^t(x)) = \Psi^t(g(x))$. If two flows are diffeomorphic, then their vector fields are related by the derivative of the conjugacy. That is, we get precisely the same result that we would have obtained if we simply transformed the coordinates in their differential equations [26].

2.4 Feature and Property Generally, let Σ denote a system with a state space $\mathcal{S}$ whose temporal evolution is captured by the flow $\Phi : \mathcal{S} \times \mathbb{R} \to \mathcal{S}$. We define an (observable) feature of Σ as a map $\mathcal{F} : \mathcal{S} \to O$ from $\mathcal{S}$ to (a possibly different) observation space O. For our system, where $\mathcal{S}$ corresponds to the product of mixed strategies of the agents $\times_i \Delta(S_i)$, typical examples of observable features are the (mixed) strategy or the (expected) utility of an agent.

The temporal evolution of system Σ induces orbits on the observation space $\Psi = \mathcal{F} \circ \Phi : \mathcal{S} \times \mathbb{R} \to O$ that encode all possible systematic interactions between the system and the $\mathcal{F}$-observer. A systematic analysis of feature $\mathcal{F}$ in Σ now translates to identifying regularities over all possible orbits of Ψ. Given feature $\mathcal{F} : \mathcal{S} \to O$, we will denote as property $\Gamma \subset O$ an open subset[4] that encodes a desirable parameter range for that specific feature.

[4]We will assume that all spaces including O are compact metric spaces.

2.5 Persistence The notions of persistence that we pursue here are inspired by more restricted notions of population persistence developed within the field of mathematical ecology[[18, 20] and references therein].

DEFINITION 2.3. *Given feature $\mathcal{F} : \mathcal{S} \to O$, let property $\Gamma \subset O$ be an open subset that encodes a desirable feature range. We state that property Γ is persistent for feature $\mathcal{F}$ if for all initial conditions $x \in int(\mathcal{S})$,we have that*

$$\liminf_{t\to\infty} dist(\mathcal{F}\big(\Phi(x,t)\big), O\backslash\Gamma) > 0.$$

Furthermore, if $\exists\ \epsilon > 0$ such that $\forall x \in int(\mathcal{S})$,we have that

$$\liminf_{t\to\infty} dist(\mathcal{F}\big(\Phi(x,t)\big), O\backslash\Gamma) > \epsilon$$

then we state that property Γ is uniformly persistent for feature $\mathcal{F}$.

These notions encode self-enforcing system regularities. That is, regardless of the starting state of the system, even if we start from states that do not satisfy a persistent property, such properties will eventually become true for the system and persist being true for all time. The typical way of enforcing regularities of similar form in a multi-agent system is to assume that the system state will converge to (a subset of) Nash equilibria and show that this property is satisfied for all such limit points. However, the notion of persistence property is stronger than these statements. For example, a property that is only true exactly at a Nash equilibrium may never be satisfied even if the system converges asymptotically to the equilibrium. On the contrary, a persistent property will eventually be satisfied. The definition of uniform persistence is even stronger and essentially states that the persistence of the property can be verified even via measurements of finite accuracy.

The are two observation spaces O of particular importance. One is the state space itself $\mathcal{S}$ in the case where $\mathcal{F}$ is the identity function. In this case, by identifying persistent properties $\Gamma \subset \mathcal{S}$ we essentially gain information about the topology of the ω-limit points of system trajectories. Nash equilibria trivially belong to this set for all dynamics that are Nash stationary (as the replicator). However, as we will see these sets can be significantly larger.

The other special case is when the observation space O is the real line $\mathbb{R}$. This is of particular importance because it allows us to forge connections to standard optimization theory. In this case, we will simply say that $\mathcal{F}$ is [m,M]-persistent if we have that: $\inf_{x\in\mathrm{int}(S)} \liminf_{t\to\infty} \mathcal{F}\big(\Phi(x,t)\big) \geq m$ and $\sup_{x\in\mathrm{int}(S)} \limsup_{t\to\infty} \mathcal{F}\big(\Phi(x,t)\big) \leq M$.

2.6 Information Theory Entropy is a measure of the uncertainty of a random variable and captures the expected information value from a measurement of the random variable. The entropy H of a discrete random variable X with possible values $\{1,\ldots,n\}$ and probability mass function $p(X)$ is defined as $H(X) = -\sum_{i=1}^{n} p(i)\ln p(i)$.

Given two probability distributions p and q of a discrete random variable their *K-L divergence* (relative entropy) is defined as $D_{\mathrm{KL}}(p\|q) = \sum_i \ln\left(\frac{p(i)}{q(i)}\right)p(i)$. It is the average of the logarithmic difference between the probabilities p and q, where the average is taken using the probabilities p. The K-L divergence is only defined if $q(i) = 0$ implies $p(i) = 0$ for all i[5]. K-L divergence is a "pseudo-metric" in the sense that for it is always non-negative and is equal to zero if and only if the two distributions are equal (almost everywhere). Other useful properties of the K-L divergence is that it is additive for independent distributions and that it is jointly convex in both of its arguments; that is, if (p_1,q_1) and (p_2,q_2) are two pairs of distributions then for any $0\le\lambda\le 1$: $D_{\mathrm{KL}}(\lambda p_1 + (1-\lambda)p_2\|\lambda q_1 + (1-\lambda)q_2) \le \lambda D_{\mathrm{KL}}(p_1,q_1) + (1-\lambda)D_{\mathrm{KL}}(p_2,q_2)$.

A closely related concept is that of the *cross entropy* between two probability distributions, which measures the average number of bits needed to identify an event from a set of possibilities, if a coding scheme is used based on a given probability distribution q, rather than the "true" distribution p. Formally, the cross entropy for two distributions p and q over the same probability space is defined as follows: $H(p,q) = -\sum_{i=1}^{n} p(i)\ln q(i) = \mathrm{H}(p) + D_{\mathrm{KL}}(p\|q)$. For more details and proofs of these basic facts the reader should refer to the classic text by Cover and Thomas [7].

3 Topology of Persistence Properties

We will start our analysis by examining the set of (uniformly) persistent properties of system trajectories. Here, we will assume that the observer $\mathcal{F}$ has full access to the realized state of the system at each time instance. In other words, the observation function $\mathcal{F}$, defined on the state space of mixed strategy outcomes $\times_i\Delta(S_i)$, is the identity function. Specifically, the existence of any uniformly persistent property implies at a minimum: $\exists\ \epsilon > 0$ such that $\forall x \in \mathrm{int}(\times_i\Delta(S_i))$:$\liminf \mathrm{dist}(\Phi(x,t), \mathrm{bd}(\times_i\Delta(S_i))) > \epsilon$.

To simplify notation, when the analysis of the orbit $\Phi(x_0,t)$, does not dependent critically on the initial starting point x_0, we will denote the state at time t, as $x(t)$ or simply x.

[5]The quantity $0\ln 0$ is interpreted as zero because $\lim_{x\to 0} x\ln(x) = 0$.

THEOREM 3.1. *Let Φ denote the flow of the replicator dynamic when applied to a network-zero-sum game and let the observation function $\mathcal{F}$ be the identity function. If $\times_i int(\Delta(S_i))$ is a uniformly persistent property of the flow then the flow has a unique interior fixed point q.*

Proof. If $\times_i\mathrm{int}(\Delta(S_i))$ is a uniformly persistent property of the flow, then by definition $\exists\epsilon > 0$ such that $\forall x \in \mathrm{int}(\times_i\Delta(S_i))$ $\liminf \mathrm{dist}(\Phi(x,t), \mathrm{bd}(\times_i\Delta(S_i))) > \epsilon$. If we denote as x_i the vector encoding the mixed strategy of agent i over her available actions at time t, then on the support of x_0 (*i.e.*, everywhere) we have that: $\int_0^t \left[u^i(R) - \sum_{R\in S_i} x_{iR}u^i(R)\right]d\tau = \int_0^t \frac{\dot{x}_{iR}}{x_{iR}}d\tau = \ln\left(\frac{x_{iR}(t)}{x_{iR}(0)}\right)$. By assumption of uniform persistence we have that for each agent i and strategy $R \in S_i$: $\liminf x_{iR} > \epsilon$ for some $\epsilon > 0$. This implies that $\lim_{t\to\infty}\frac{1}{t}\ln\left(\frac{x_i(t)}{x_i(0)}\right) = 0$. For any pair of agent i and strategy R, the functions $\frac{1}{t}\int_0^t x_{iR}d\tau$, $\frac{1}{t}\int_0^t u^i(R)d\tau$ are bounded. Since they are finitely many of them we can find a common converging subsequence t_n for all of them[6]. Combining the last two equations and dividing them with t_n we derive for every agent i, $R \in S_i$:

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^t \sum_{R\in S_i} x_{iR}u^i(R)d\tau = \lim_{n\to\infty}\frac{1}{t_n}\int_0^t u^i(R)d\tau =$$

$$= \lim_{n\to\infty}\frac{1}{t_n}\int_0^t \mathrm{E}_{s_{-i}\sim x_{-i}(\tau)}u_i(R,s_{-i})d\tau = u_i(R,\hat{x}_{-i})$$

where $\hat{x}_{iR} = \lim_{n\to\infty}\frac{1}{t_n}\int_0^t x_{iR}d\tau$ and the last equation follows from the separability of payoffs. Since for all agents i, $\forall R,Q \in S_i : u_i(R,\hat{x}_{-i}) = u_i(Q,\hat{x}_{-i})$, $\hat{x}$ is a fully mixed Nash equilibrium.

Let's assume that the system has two distinct interior fixed points. These correspond to fully mixed equilibria q_1,q_2. By linear separability of payoffs any point of the state space of mixed strategy profiles that lies on the (infinite) line connecting q_1,q_2 is also a fixed point. However, this line hits the boundary and as a result, for any $\epsilon > 0$ we can find interior fixed points of the system at distance less than ϵ from the boundary. We reach a contradiction, since we have assumed that there should exist an $\epsilon > 0$ such that $\forall x \in \mathrm{int}(\times_i\Delta(S_i))$ we have that $\liminf \mathrm{dist}(\Phi(x,t), \mathrm{bd}(\times_i\Delta(S_i))) > \epsilon$. $\square$

We will show that the cross entropy between a fully mixed Nash q and an evolving interior state $\sum_i\sum_{R\in S_i} q_{iR}\cdot\ln(x_{iR})$ is an invariant of the dynamics. When $x,y \in \times_i\Delta(S_i)$ we will use $H(x,y), D_{\mathrm{KL}}(x,y)$ to denote respectively the $\sum_i H(x_i,y_i), \sum_i D_{\mathrm{KL}}(x_i,y_i)$.

[6]Take a convergent subsequence of the first function and find on this a convergence subsequence of the second and so on.

THEOREM 3.2. *Let Φ denote the flow of the replicator dynamic when applied to a network-zero-sum game that has an interior (i.e. fully mixed) Nash equilibrium q then given any (interior) starting point $x_0 \in \times_i \Delta(S_i)$, the cross entropy between q and the state of system $\Phi(x_0, t)$ is a constant of the motion, i.e., it remains constant as we move along any system trajectory.*

Otherwise, let q be a (not fully mixed) Nash equilibrium of the game on $bd(\times_i \Delta(S_i))$, then for each starting point $x_0 \in \times_i int(\Delta(S_i))$ for all $t' \geq 0$ $\frac{dH(q,\Phi(x_0,t))}{dt}|_{t=t'} < 0$.

Proof. The support of the state of system (*e.g.,* the strategies played with positive probability) is an invariant of the flow, so it suffices to prove this statement for each starting point x_0 at time $t = 0$. We examine the derivative of $H(q, \Phi(x_0, t)) = -\sum_i \sum_{R \in S_i} q_{iR} \cdot \ln(x_{iR})$.

$$\begin{aligned}
\sum_i \sum_{R\in S_i} q_{iR} \frac{d\ln(x_{iR})}{dt} &= \sum_i \sum_{R \in S_i} q_{iR} \frac{\dot{x}_{iR}}{x_{iR}} = \\
&= \sum_i \sum_{(i,k)\in E} \left(q_i^{\mathrm{T}} A^{i,k} x_k - x_i^{\mathrm{T}} A^{i,k} x_k\right) = \\
&= \sum_i \sum_{(i,k)\in E} \left(q_i^{\mathrm{T}} - x_i^{\mathrm{T}}\right) A^{i,k} x_k \geq \\
&\geq \sum_i \sum_{(i,k)\in E} \left(q_i^{\mathrm{T}} - x_i^{\mathrm{T}}\right) A^{i,k} (x_k - q_k) = \\
&= \sum_{E=(i,k)} \big[\left(q_i^{\mathrm{T}} - x_i^{\mathrm{T}}\right) A^{i,k} (x_k - q_k) + \\
&\quad + \left(q_k^{\mathrm{T}} - x_k^{\mathrm{T}}\right) A^{k,i} (x_i - q_i)\big] = 0.
\end{aligned}$$

For each agent i, $\sum_{(i,k)\in E} \left(q_i^{\mathrm{T}} - x_i^{\mathrm{T}}\right) A^{i,k} q_k \geq 0$, since q is a Nash equilibrium. Since the state x is fully mixed, $\sum_i \sum_{(i,k)\in E} \left(q_i^{\mathrm{T}} - x_i^{\mathrm{T}}\right) A^{i,k} q_k = 0$ if and only if the Nash equilibrium q is fully mixed. □

The cross entropy between the Nash q and the state of the system, however is equal to the summation of the K-L divergence between these two distributions and the entropy of q. Since the entropy of q is constant, we derive the following corollary:

COROLLARY 3.1. *If the flow Φ has an interior fixed point q then given any (interior) starting point $x_0 \in \times_i \Delta(S_i)$, the K-L divergence between q and the state of the system is a constant of the motion.*

So far, we have shown that for a system to have a uniformly persistent property it must have a (unique) fully mixed Nash equilibrium. Furthermore, for games with a fully mixed Nash equilibrium the K-L divergence (between that equilibrium and the state of the system) remains constant. These replicator flows stay bounded away from the boundary. In this case, we will show how to apply Poincaré recurrence to establish the recursive nature of the flow.

THEOREM 3.3. *If the flow Φ has an interior fixed point, then for each open set E that is bounded away from $bd(\times_i \Delta(S_i))$ there exist orbits that intersect E infinitely often.*

Proof. Let q be the interior fixed point of the flow. By corollary 3.1, we have established in this case, the K-L divergence between q and the state of the system is a constant of the motion. This implies that starting off any interior point the system trajectory will always remain bounded away from the boundary[7]. Furthermore, the system defined by applying replicator on the interior of state space, can be transformed to a divergence free system on $(-\infty, +\infty)^{\sum_i (|S_i|-1)}$ via the following invertible smooth map $z_{iR} = \ln(x_{iR}/x_{i0})$, where $0 \in S_i$ a specific (but arbitrarily chosen) strategy of agent i. This map $g : \times_i \mathrm{int}(\Delta(S_i)) \to \mathbb{R}^{\sum_i (|S_i|-1)}$ is clearly a homeomorphism[8]. Hence, we can a establish a conjugacy between the replicator system (restricted to the interior of state space) and a system on $(-\infty, +\infty)^{\sum_i (|S_i|-1)}$ where:

$$\frac{d\left(\frac{x_{iR}}{x_{i0}}\right)}{dt} = \frac{\dot{x}_{iR} x_{i0} - \dot{x}_{i0} x_{iR}}{x_{i0}^2} = \frac{x_{iR}}{x_{i0}} \big(u^i(R) - u^i(0)\big).$$

This implies that $\dot{z_{iR}} = \frac{d\left(\ln \frac{x_{iR}}{x_{i0}}\right)}{dt} = u^i(R) - u^i(0)$ where $u^i(R), u^i(0)$ depend only on the mixed strategies of the rest of the agents (i.e. other than i). As a result, the flow $\Psi^t = g \circ \Phi^t \circ g^{-1}$, which arises from our system via the change of variables $z_{iR} = \ln(x_{iR}/x_{i0})$, defines a separable vector field in the sense that the evolution of z_{iR}, depends only on the state variables of the other agents. The diagonal of the Jacobian of this vector field is zero and consequently the divergence (which corresponds to the trace of the Jacobian) is zero as well. Liouville's theorem states that such flows are volume preserving. On the other hand, this transformation blows up the volume near the boundary to infinity and as a result does not allow for an immediate application of Poincaré's recurrence theorem.

[7] This statement follows from the fact that the K-L divergence becomes infinite at the boundary, whereas it is finite for any interior (starting) point.

[8] The reverse map is $x_{i0} = \frac{1}{1+\sum_{i \in S_i \setminus \{0\}} e^{z_{iR}}}$, $x_{iR} = \frac{e^{z_{iR}}}{1+\sum_{R \in S_i \setminus \{0\}} e^{z_{iR}}}$ for $R \in S_i \setminus \{0\}$. In fact, g is a diffeomorphism.

Given any open set E that is bounded away from the boundary and let $c_E = \sup_{x \in E} D_{\mathrm{KL}}(q\|x)$. Since E is bounded away from the boundary c_E is finite. We focus on the restriction of flow Ψ over the closed and bounded set[9] $g(\mathcal{S}_E)$, where $\mathcal{S}_E = \{x \in \times_i \Delta(S_i) : D_{\mathrm{KL}}(q\|x) \leq c_E\}$. The fact that replicator preserves K-L divergence between the equilibrium q and the state of the system implies that replicator maps $\mathcal{S}_E$ to itself. Due to the homeomorphism g, the same applies for flow Ψ and $g(\mathcal{S}_E)$. The restriction of flow Ψ on $g(\mathcal{S}_E)$ is a volume preserving flow and has only bounded orbits. As a result, we can now apply Poincaré's theorem to derive that for each open set of this system, there exist orbits $\Psi(z_0, \cdot)$ that intersect the set infinitely often. Given our initial arbitrary (but bounded from the boundary of $\times_i \Delta(S_i)$)) open set E, $g(E)$ is also open[10] and hence infinitely recurrent for some $\Psi(z_0, \cdot)$ but now the $g^{-1}(\Psi(z_0, \cdot)) = \Phi(g^{-1}(z_0), \cdot)$ visits E infinitely often, concluding the proof. $\square$

Combining the theorems in this section we derive the following characterization of persistent properties.

COROLLARY 3.2. *Φ has no uniformly persistent property $\Gamma \subset \times_i int(\Delta(S_i))$. If the flow has an interior fixed point then $\times_i int(\Delta(S_i))$ is a persistent property of the system, however, any property $\Gamma \subset \times_i \Delta(S_i)$ whose complement $\times_i \Delta(S_i) \backslash \Gamma$ contains a ball of radius $\epsilon > 0$ is not persistent.*

3.1 Systems without interior fixed points As shown by [6], the set of Nash equilibria for separable zero-sum games is convex. Such games exhibit a unique maximal support of Nash equilibrium strategies.[11] We will denote by $W_i \subset S_i$ the support of agent's i mixed strategy in any of the maximum support size equilibria. As we have argued, when there exist fully mixed Nash equilibria the only persistent property is the set $\times_i \mathrm{int}(\Delta(S_i))$. We will show that as the size of W_i decreases, the predictability of the system increases.

THEOREM 3.4. *If Φ does not have an interior fixed point, then given any interior starting point $x \in int(\times_i \Delta(S_i))$, the orbit $\Phi(x, \cdot)$ converges to the boundary of the state space. Furthermore, if q is an equilibrium of maximum support with W_i the respective supports of each agent's strategy then $\omega(x) \subset \times_i int(\Delta(W_i))$.*

In simple terms, our analysis implies that the limit sets of systems without interior fixed points correspond to limit sets of a collapsed subsystem which assigns probability zero to any strategy that lies outside the maximum equilibrium support. This allows for a unified treatment of all replicator flows on separable zero-sum games by focusing on the right subspace defined by the maximal Nash equilibrium support. Furthermore, we derive a general information theoretic principle which implies that the limit behavior of all such flows satisfies a universal information theoretic conservation law.

COROLLARY 3.3. ***Information Conservation in the Limit:*** *Let q be a Nash equilibrium of maximum support, then the rate of change of the K-L divergence between q and the state of the system converges to zero.*

Next, we will reinterpreting these characterization results about the system limit sets from an optimization perspective. Specifically, we will use them to identify in polynomial time accurate estimators about the extremal values of different features $\mathcal{F} : \times_i \Delta(S_i) \to \mathbb{R}$ of the system state.

4 Features and Optimization

The take-home message of the topological investigation of system trajectories is that the long-run behavior of the system is dictated by the maximum support Nash equilibria of the separable zero-sum game. If we can compute these maximum supports W_i for each agent i efficiently then this defines convex subspaces over which we can apply optimization techniques. We extend tools from Cai and Daskalakis [6] to find these supports in polynomial time.

THEOREM 4.1. *Given a separable constant-sum multiplayer game we can find a Nash equilibrium of maximum support in polynomial time.*

We will show that by performing optimization over this product of simplices we can find the possible range of continuous features $\mathcal{F}$ of our system regardless of initial conditions.

THEOREM 4.2. *Let $\mathcal{F} : \times_i \Delta(S_i) \to \mathbb{R}$ be a continuous feature of the state space and let W_i be the support of agent i in a maximum support equilibrium then we have that:*

$$\sup_{x \in \times_i int(\Delta(S_i))} \limsup_{t \to \infty} \mathcal{F}(\Phi(x, t)) \leq \max_{x \in \times_i \Delta(W_i)} \mathcal{F}(x).$$

[9] $g(\mathcal{S}_E)$ is closed since $\mathcal{S}_E$ is closed and g is a homeomorphism. $g(\mathcal{S}_E)$ is bounded since E is bounded away from the boundary of $\times_i \Delta(S_i)$.

[10] Since g is a homeomorphism.

[11] Indeed, if there exist two equilibria of maximal support with distinct supports, then the mixed strategy profile where each agent chooses uniformly at random to follow his randomized strategy in one of the two distributions is also a Nash equilibrium and has larger support that each of the original distributions.

Proof. We will prove that for each $x \in \times_i \mathrm{int}(\Delta(S_i))$: $\limsup_{t\to\infty} \mathcal{F}(\Phi(x,t)) \le \max_{x\in\times_i\Delta(W_i)} \mathcal{F}(x)$. By theorem 3.4 we have that given any $x_0 \in \times_i \mathrm{int}(\Delta(S_i))$: $\omega(x_0) \subset \times_i \mathrm{int}(\Delta(W_i))$. Let's assume that there exists $x \in \times_i \mathrm{int}(\Delta(S_i))$: $\limsup_{t\to\infty} \mathcal{F}(\Phi(x,t)) > \max_{x\in\times_i\Delta(W_i)} \mathcal{F}(x)$. This implies that there exists sequence $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \mathcal{F}(\Phi(x_0,t_n)) > \max_{x\in\times_i\Delta(W_i)} \mathcal{F}(x)$. Therefore, this sequence $\Phi(x_0,t_n)$ must exhibit a subsequence converging to a point $z \in \times_i\Delta(S_i) \setminus \times_i\Delta(W_i)$. We have reached a contradiction, since $\forall x_0 \in \times_i \mathrm{int}(\Delta(\mathcal{S}_i))$: $\omega(x_0) \subset \times_i \mathrm{int}(\Delta(W_i))$. □

We can also apply our topological characterization results to identify analogous lower bounds.

THEOREM 4.3. *Let $\mathcal{F} : \times_i\Delta(S_i) \to \mathbb{R}$ be a continuous feature of the state space and let W_i be the support of agent i in a maximum support equilibrium then we have that:*

$$\sup_{x\in\times_i int(\Delta(W_i))} \limsup_{t\to\infty} \mathcal{F}(\Phi(x,t)) \ge \max_{x\in\times_i\Delta(W_i)} \mathcal{F}(x).$$

Proof. Given any point $x \in \times_i\Delta(W_i)$ and for any $\delta > 0$, we can create an open set $S_x^\delta \subset \times_i \mathrm{int}(\Delta(W_i))$ bounded away from the boundary such that $\sup_{y\in S_x^\delta} \|x-y\|_2 \le \delta$ and $\inf_{y\in S_x^\delta} \|x-y\|_2 \ge \delta/2$. As we have argued in theorem 3.3 such sets are infinitely recurrent. Given any such set S_x^δ there exists a $y(x,\delta)$ and $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \Phi(y(x,\delta),t_n) \in S_x^\delta$. Since $\mathcal{F} : \times_i\Delta(S_i) \to \mathbb{R}$ is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \times_i\Delta(S_i)$ with $\|x-y\|_2 < \delta$ we have that $|\mathcal{F}(x) - \mathcal{F}(y)| < \epsilon$. Putting all this together, we have that for any $x \in \arg\max_{x\in\times_i\Delta(W_i)} \mathcal{F}(x)$ and any $\epsilon > 0$ there exists a $y \in \mathrm{int}(\Delta(W_i))$: $\limsup_{t\to\infty} \mathcal{F}(\Phi(y,t)) > \mathcal{F}(x) - \epsilon$. □

Naturally, we can derive analogous relations for inf and min. Combining the above two theorems we derive the following tight characterization:

COROLLARY 4.1. *Let Φ the replicator flow when applied to a separable constant-sum multiplayer game that has an interior Nash equilibrium and $\mathcal{F} : \times_i\Delta(S_i) \to \mathbb{R}$ a continuous feature of the state space then*

$$\sup_{x\in\times_i int(\Delta(S_i))} \limsup_{t\to\infty} \mathcal{F}(\Phi(x,t)) = \max_{x\in\times_i\Delta(S_i)} \mathcal{F}(x).$$

It should be clear that if we combine the results of this section with "nice" features $\mathcal{F}$ (e.g. a linear functions of the state space), then we can identify tight bounds on the extremal values of such features over all system orbits efficiently. We provide some explicit applications along those lines in Appendix A.

5 Concluding Remarks

We develop a novel framework to address analytical challenges that arise in decentralized systems. Our approach does not prescribe a "correct" equilibrium concept. On the contrary, our goal is to embrace the true complexity of such systems and to identify their persistent properties efficiently. Along the way we bring together tools and intuitions from traditionally disparate but well developed areas such as theory of optimization, topology of dynamical systems, algorithmic game theory, as well as, novel ideas. Our results go against established intuitions about chaotic environments and the obstacles they pose to algorithmic inquiry. Exploring the limits of the applicability of these ideas is a rather promising area for future work.

Specifically, we believe that from this starting point two interesting challenges spring forward. One is a question about the analysis of socioeconomic environments, where the other is about the efficient design of complex systems.

The replicator dynamics as a model of actual human decision making is overly simplistic and hopelessly incomplete. So, is any other model. Nevertheless, system analysis in game theoretic settings can still be rather useful. This is because it can be indicative about which is the natural mathematical language in which real life solutions should be framed. The established practice in game theory is to output solution points (specifically Nash equilibria). This work indicates that this approach is too restrictive even for the flagship case of zero-sum games. *The descriptive framework of Nash equilibria does not suffice to capture critical subtleties of these environments.* Given the importance of zero-sum games (as well as their natural network extensions) we should possibly consider more flexible set-based solution concepts. The idea of persistence patterns (adapted from mathematical ecology) attempts a step in this direction, and we hope that future work will explore these ideas further.

In the reverse direction, when it comes to designing large decentralized artificial systems we have complete control over the operational dynamics. The choice of dynamics is a key design decision. Replicator dynamics (and its variants) could act here as a catalyst in terms of establishing novel design paradigms. The connections that replicator dynamics effortlessly weaves between information theory, physics, computer science, topology, biology, ecology and control theory compose a potent mix that calls to be put into action. We need to see replicator trajectories for what they really are; a powerful, naturally occurring, computational primitive and then understand how to exploit them. Exploratory steps in this direction [27] are currently under way.

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A Applications

As we have shown we can compute the maximum Nash equilibrium support W_i for each agent i efficiently. Typically, a constant-sum economy captures the competition between two of more service providers that vie for the attention of as large a subsection of the customer base as possible. We normalize our constant-sum games so that the sum of utilities of the agents is equal to 1 and that the agent utilities are always positive. The utility of an agent captures its market share. In this case, every market participant is faced with two key questions:

- Will my company survive in the long run?
- Is my company achieving its full potential?

We will show how can we apply our framework to provide insights on this kind of question.

A.1 Margins of Survival In this case, the feature $\mathcal{F}$ is the utility of each agent u_i. In terms of the survivability of each company, we can assume that each company i has some margin of survival γ_i. If its market share drops below this critical threshold γ_i then there is no turning back and the company is doomed. The company's survival can only be guaranteed, regardless of the initial condition x, as long as $\inf_{x\in\times_i \mathrm{int}(\Delta(S_i))} \liminf_{t\to\infty} u_i(\Phi(x_0,t)) \geq \gamma_i$. Going back to proposition 4.2, we have shown the following: $\inf_{x\in\times_i \mathrm{int}(\Delta(S_i))} \liminf_{t\to\infty} u_i(\Phi(x_0,t)) \geq \min_{x\in\times_i\Delta(W_i)} u_i(x)$. The (expected) utility of the agent over a randomized strategy is defined via a multilinear extension of the values of the utility function over the pure strategy outcomes $x \in \times_i W_i$. In this case, we have that $\min_{x\in\times_i\Delta(W_i)} u_i(x) = \min_{x\in\times_i W_i} u_i(x)$ and the latter problem can be trivially solved in polynomial time.

A.2 Optimal Market Performance Once again, the relevant feature $\mathcal{F}$ is the utility of each agent u_i. The optimal market share that a company can ever hope to achieve, once the initial transients of the market have settled down is captured by the term $\sup_{x\in\times_i \mathrm{int}(\Delta(S_i))} \limsup_{t\to\infty} u_i(\Phi(x_0,t))$. Applying again proposition 4.2, we have the following: $\sup_{x\in\times_i \mathrm{int}(\Delta(S_i))} \limsup_{t\to\infty} u_i(\Phi(x_0,t)) \leq \max_{x\in\times_i\Delta(W_i)} u_i(x)$. Similarly, as in the case of the minimum we can compute this maximum in polynomial time.

B Proof of Theorem 3.4

We commence the analysis with the following technical lemma, whose proof can be found in [1] but is also provided here for completeness:

Lemma B.1. *If $g(t)$ is a twice differential function with uniformly bounded second derivative and $\lim_{t\to\infty} g(t)$ exists and is finite then we have that $\lim_{t\to\infty} \dot{g}(t) = 0$.*

Proof. Let's denote by $M \geq 1$ an upper bound on the second derivative of g. Suppose that the statement was not true. In this case, we would be able to find a sequence $\{t_n\}$ going to infinity such that $\dot{g}(t_n)$ remains bounded away from zero. Next, we can assume that $t_{n+1} > t_n + 1$ and $\dot{g}(t_n) \geq \epsilon$ for some $0 < \epsilon < 1$. If we define $g_{2n} = g(t_n)$ and $g_{2n+1} = g(t_n + \frac{\epsilon}{2M})$, then a first application of the mean value theorem implies that $\dot{g}(t) \geq \frac{\epsilon}{2}$ for $t_n \leq t \leq t_n + \epsilon/2M$. A second application implies that $g_{2n+1} - g_{2n} \geq \frac{\epsilon^2}{4M} > 0$. Hence, $\lim_{n\to\infty} g_n$ if it exists, is infinity. Therefore, the same holds for $\lim_{t\to\infty} g(t)$. □

We prove the following asymptotic property of the orbits of flow Φ.

Theorem B.1. *Let Φ denote the flow of the replicator dynamic when applied to a network-zero-sum game. If Φ does not exhibit an interior fixed point, then given any interior starting point $x \in int(\times_i\Delta(S_i))$, the orbit $\Phi(x,\cdot)$ converges to the boundary of the state space. Furthermore, if q is an equilibrium of maximum support with W_i the respective supports of each agent's strategy then $\omega(x) \subset \times_i int(\Delta(W_i))$.*

Proof. From the second part of proposition 3.2, we have that starting from any fully mixed strategy profile $x_0 \in \times_i \mathrm{int}(S_i)$ and for all $t' \geq 0$ $\frac{dH(q,\Phi(x_0,t))}{dt}|_{t=t'} < 0$. However, $D_{\mathrm{KL}}(q\|\Phi(x(0),t))$ is bounded below at zero, and since it is strictly decreasing it must exhibit a finite limit. Since $\frac{dH(q,\Phi(x_0,t))}{dt} = -\sum_i \sum_{(i,k)\in E} q_i^{\mathrm{T}} A^{i,k} x_k$, it is immediate that $D_{\mathrm{KL}}(q\|\Phi(x_0,t))$ has bounded second derivatives. Lemma B.1 implies that $\lim_{t\to\infty} \frac{dH(q,\Phi(x_0,t))}{dt} = 0$. This implies that for any $y \in \omega(x_0)$ we have that the support of y must be a subset of the support of q. The support of q is not complete in the first place since q is not a fully mixed equilibrium, hence y must lie on the boundary as well. Finally, since $D_{\mathrm{KL}}(q\|\Phi(x_0,t))$ has a finite limit each $y \in \omega(x(0))$ must assign positive probability to all strategies in the support of q. □

C Proof of Theorem 4.1

Here, we will use a slightly different notation for n-person separable zero-sum multiplayer games, following that of [6] since it helps with reducing the notational burden for this section and it allows us to apply verbatim results from that paper. Let $GG \triangleq \{A^{u,v}, A^{v,u}\}$ be an n-person separable zero-sum multiplayer game. Every agent u has m_u strategies. We set $A^{u,v} = A^{v,u} = 0$ for all pairs $(u,v) \notin E$. Let the corresponding lawyer game $G = (R,C)$ be a symmetric $\sum_u m_u \times \sum_u m_u$ bimatrix game, whose rows and columns are indexed by pairs $(u:i)$ of agents $u \in [n]$ and strategies $i \in [m_u]$. For all $u,v \in [n]$ and $i \in [m_u], j \in [m_v]$ we have that:

$$R_{(u:i),(v:j)} = A^{u,v}_{i,j} \text{ and } C_{(u:i),(v:j)} = A^{v,u}_{j,i}$$

Intuitively, the formulation is such that each lawyer can choose a strategy belonging to any one of the nodes of GG. If they happen to choose strategies of adjacent nodes, they receive the corresponding payoffs that the nodes would receive in GG from their joint interaction.

In [6], Cai and Daskalakis show that finding a (generic) Nash equilibrium of a separable constant-sum game is equivalent to solving the following LP:

$$
\text{(C.1)} \qquad \begin{aligned} & \max \tfrac{1}{n} \textstyle\sum_u \hat{z}_u \\ \text{s.t.} \quad & x^T \cdot R \geq z^T; \\ & z_{u:i} = \hat{z}_u \forall u, i; \\ & \textstyle\sum_{i \in [m_u]} x_{u:i} = \tfrac{1}{n}, \quad \forall u; \\ & x_{u:i} \geq 0, \qquad \forall u, i. \end{aligned}
$$

We define the following bijection between vectors x satisfying $\sum_{i \in [m_u]} x_{u:i} = \frac{1}{n}, \forall u; x_{u:i} \geq 0, \forall u, i$ and mixed strategies y of game GG as following: $\forall u \in [n], \forall i \in [m_u]$ we set $y_u(i) = n \cdot x_{u:i}$. Specifically, we have that:

Theorem C.1. *[6] Given an optimal solution to the LP (C.1), the set of vectors y_u with $y_u(i) = n \cdot x_{u:i}, \forall u \in [n], \forall i \in [m_u]$ is a Nash equilibrium of the separable zero-sum game GG. Given any Nash equilibrium $y_1, y_2, \ldots, y_n$ of the separable zero-sum game there exists an optimal solution of the LP (C.1), with $x_{u:i} = \frac{y_u(i)}{n}, \ \forall u \in [n], \forall i \in [m_u]$ for some $z, \hat{z}$. The optimal value of the LP is zero for all separable zero-sum games. The set of Nash equilibria is convex.*

As we have already argued, since the set of Nash equilibria is convex, all equilibria of maximal support have exactly the same support (for all agents). We will find this maximum support in polynomial time. Moreover, we will find an equilibrium of maximum support. Although finding such an equilibrium is NP-hard [15] in general-sum games, we show that its computation is tractable in the case of separable constant-sum multiplayer games. Exploring these issues in other classes of games could be an interesting direction for future work.

Theorem C.2. *Given a separable constant-sum multiplayer game we can find a Nash equilibrium of maximum support in polynomial time.*

Proof. Since the optimal value of the lawyer LP (C.1) has optimal value equal to zero then any optimal solution to LP is a feasible solution to the linear feasibility problem defined by the adding the linear constraint $\sum_u \hat{z}_u \geq 0$. to the constraints of the (C.1). The reverse is also trivially true. Incorporating theorem (C.1) we have that given a solution to our linear feasibility program, the set of vectors y_u with $y_u(i) = n \cdot x_{u:i} \forall u \in [n], \forall i \in [m_u]$ encodes a Nash equilibrium of the separable zero-sum game GG. Given any Nash equilibrium $y_1, y_2, \ldots, y_n$ of the separable zero-sum game there exists a feasible solution of our linear feasibility program with with $x_{u:i} = \frac{y_u(i)}{n}, \ \forall u \in [n], \forall i \in [m_u]$ for some $z, \hat{z}$. Given agent $u \in [n]$ and $i \in [m_u]$, we define the following LP:

$$
\text{(C.2)} \qquad \begin{aligned} & \max x_{u:i} \\ \text{s.t.} \quad & \textstyle\sum_u \hat{z}_u \geq 0 \\ & x^T \cdot R \geq z^T; \\ & z_{u:i} = \hat{z}_u \forall u, i; \\ & \textstyle\sum_{i \in [m_u]} x_{u:i} = \tfrac{1}{n}, \quad \forall u; \\ & x_{u:i} \geq 0, \qquad \forall u, i. \end{aligned}
$$

The x vector in an optimal solution of the LP above pinpoints out of the set of all Nash equilibria of game GG, an equilibrium where the probability that agent u assigns to strategy i is maximal. In other words, the value of the LP above is positive if and only if there exists a Nash equilibrium where agent u assigns positive probability to action i. We find an optimal solution for each LP of the form C.2 for all agents $u \in [n]$ and $i \in [m_u]$. We can do this efficiently since there exist only polynomially many such combinations. Finally, since the set of Nash equilibria is convex, we output the probability distribution that corresponds to each agent u choosing amongst his respective mixed strategies in each of the solutions of the LPs above uniformly at random. By convexity of the set of Nash equilibria this strategy profile is also a Nash equilibrium.

Finally, this Nash equilibrium is of maximum support. Let's assume that there existed a Nash equilibrium in which an agent v assigned positive probability to a strategy j while in our defined mixture we get $x_{v:j} = 0$. However, the optimal value of the linear program C.2 corresponding to v, j would be greater than zero, reaching a contradiction about $x_{v:j} = 0$ in our mixture of the LP solutions. $\square$